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Memoir on Symmetric Functions of the Roots of Systems of Equations

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VII. *Memoir on Symmetric Functions of the Roots of Systems of Equations.*

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§ 1. *Preliminary Ideas.*

1. THE theory of the symmetrical functions of a single system of quantities has been investigated in a large number of memoirs, but so far, only a few attempts have been made to develop an analogous theory with regard to several systems of quantities. The chief authors are SCHLÄFLI * and CAYLEY,† both of whom have, however, restricted themselves to the outlines of the commencement of such a theory. In the theory of the single system it is found convenient to regard the quantities as the roots of an equation, since the coefficients of such an equation are themselves those particular symmetric functions of the quantities which have been variously termed fundamental, elementary, and unitary; they are fundamental because all other rational integral functions are expressible by their products of the same or lower degree; elementary because they are those which, first of all, naturally arise; unitary because their partitions are composed wholly of units. The left hand side of the equation referred to is a product of binomial linear functions of a single variable x, so that, $\alpha_1, \alpha_2, \ldots \alpha_n$ being the quantities which compose the system, the fundamental relation may be written

$$(1+\alpha_1 x)(1+\alpha_2 x)\ldots(1+\alpha_n x) = 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n ,$$
$$= 1 + (1)x + (1^2)x^2 + \ldots + (1^n)x^n ,$$

in the ordinary partition notation.

In a general discussion it is convenient and advantageous to suppose the number of quantities infinite, so that the relation becomes

$$(1+\alpha_1 x)(1+\alpha_2 x)\ldots = (1 + a_1 x + a_2 x^2 + \ldots) = 1 + (1)x + (1^2)x^2 + \ldots$$

* "Ueber die Resultante eines Systemes mehrerer algebraischen Gleichungen." 'Vienna Academy *Denkschriften*,' vol. 4, 1852.

† "On the Symmetric Functions of the Roots of certain Systems of Two Equations." 'Phil. Trans.,' vol. 147 (1857).

2. Instead of taking a product of binomial linear functions of one variable, as above, we can, for m systems of quantities, take a product of non-homogeneous linear functions of m variables, and each such linear function may be taken of the form

$$1 + \alpha_{s1}x_1 + \alpha_{s2}x_2 + \ldots + \alpha_{sm}x_m.$$

As indicative of this general case it is sufficient to consider merely the case of two systems of quantities. Complexity of formulas is thereby avoided, but it must be distinctly borne in mind that all the succeeding theorems can be at once extended to the general case of m systems by an easy enlargement of the nomenclature and notation.

I consider, then, two systems of quantities

$$\alpha_1, \ \alpha_2, \ \ldots \ \alpha_n;$$
$$\beta_1, \ \beta_2, \ \ldots \ \beta_n;$$

as connected with two non-homogeneous equations, in two variables, in such wise that the values α_s, β_s, of the variables respectively constitute one solution of the two simultaneous equations. In order to avoid identical relations between fundamental forms, as well as for other reasons, which will appear, I take the number of quantities n in each system to be infinite.

By analogy the fundamental relation is written

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_s x + \beta_s y) \ldots$$
$$= 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots + a_{pq}x^p y^q + \ldots$$

As shown by SCHLÄFLI this equation may be directly formed and exhibited as the resultant of the two given equations, and an arbitrary, linear, non-homogeneous equation in two variables. Beyond the preliminary idea this investigation has little to do with the original equations or with the theory of resultants. It starts with the fundamental equation just written, the right-hand side of which may be put into the form

$$1 + \Sigma\alpha_1 x + \Sigma\beta_1 y + \Sigma\alpha_1\alpha_2.x^2 + \Sigma\alpha_1\beta_2.xy + \Sigma\beta_1\beta_2.y^2 + \ldots$$

The most general symmetric function to be considered is

$$\Sigma\alpha_1^{p_1}\beta_1^{q_1}\alpha_2^{p_2}\beta_2^{q_2}\alpha_3^{p_3}\beta_3^{q_3}\ldots$$

which I represent symbolically by

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\ldots).$$

Observe that the summation is in regard to the expressions obtained by permuting the n suffices

$$1, 2, 3, \ldots n.$$

The weight of the function must be considered as bipartite; it consists of the two numbers

$$p_1 + p_2 + p_3 + \ldots = \Sigma p,$$
$$q_1 + q_2 + q_3 + \ldots = \Sigma q,$$

and I speak of the biweight Σp, Σq.

The sum $\Sigma p + \Sigma q$ may be called the whole weight, or simply the weight. Associated with any number w there will be a weight w and a biweight corresponding to every composition of w by means of two numbers, including zero as a number. By composition is meant partition, in which regard is paid to the order of the parts; for instance, 21 and 12 are different binary compositions of 3, and 30, 21, 12, 03 constitute the system.

3. It is necessary to introduce the notion of the partition of the bipartite number which denotes the biweight.

Thus of the biweight Σp, Σq the expression

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\ldots)$$

may be termed a partition.

The dual symbols $\overline{p_1q_1}$, $\overline{p_2q_2}$, $\overline{p_3q_3}$, ... are the parts of this partition; the parts are themselves bipartite and may be termed biparts.

We have thus a biweight denoted by a bipartite number partitioned into a number of bipartite numbers termed biparts.

It is convenient to arrange the biparts so that the sums of the symbols which compose them are in descending order of magnitude from left to right.

According to usual practice repetitions of biparts are denoted by power symbols; thus

$$(\overline{p_1q_1}^2) \equiv (\overline{p_1q_1}\,\overline{p_1q_1}).$$

4. In the notation just explained the fundamental relation is written

$$\begin{aligned}(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y)\ldots \\ = 1 + (\overline{10})\,x + (\overline{01})\,y + (\overline{10}^2)\,x^2 + (\overline{10}\,\overline{01})\,xy + (\overline{01}^2)\,y^2 \\ + (\overline{10}^3)\,x^3 + (\overline{10}^2\,\overline{01})\,x^2y + (\overline{10}\,\overline{01}^2)\,xy^2 + (\overline{01}^3)\,y^3 + \ldots\end{aligned}$$

where $(\overline{10}\,\overline{01}^2)$ denotes $\Sigma\,\alpha_1\,\beta_2\,\beta_3$ and in general $a_{pq} = (\overline{10}^p\,\overline{01}^q)$.

Observe that here the number of quantities in each system is considered to be infinite, and that the right hand side of the equation is taken with unit and not multinomial coefficients (*cf.* CAYLEY, *loc. cit.*). This is done because it is the universal practice in the theory of the single system, and because otherwise it appears to possess undoubted advantages.

The symmetric functions which appear in the relation are fundamental since, as will appear, they serve to express all other rational integral symmetric functions, and they may be further termed single-unitary, in that, not only is each composed entirely of units, but also each bipart comprises but a single unit.

It is obvious that the number of biweights connected with the weight w is $w+1$.

5. It may be asked in how many ways it is possible to partition a biweight into biparts.

In the ordinary theory of partitions the number of partitions of a number w is the coefficient of x^w in the ascending expansion of

$$\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3\,.\,1-x^4\ldots}.$$

In the present case, the number of partitions of the biweight pq into biparts is the coefficient of x^py^q in the ascending expansion of

$$\frac{1}{1-x\,.\,1-y\,.\,1-x^2\,.\,1-xy\,.\,1-y^2\,.\,1-x^3\,.\,1-x^2y\,.\,1-xy^2\,.\,1-y^3\ldots},$$

or, putting y equal to x, we see that the whole number of partitions of the weight $p+q$ into biparts is the coefficient of x^{p+q} in the ascending expansion of

$$\frac{1}{(1-x)^2\,(1-x^2)^3\,(1-x^3)^4\ldots}.$$

Further, it is clear that the number of partitions of the biweight pq into exactly μ biparts is the coefficient of $a^\mu x^p y^q$ in the expansion of

$$\frac{1}{1-ax\,.\,1-ay\,.\,1-ax^2\,.\,1-axy\,.\,1-ay^2\,.\,1-ax^3\,.\,1-ax^2y\,.\,1-axy^2\,.\,1-ay^3\ldots}.$$

6. It is convenient now to have before us a list of the symmetric functions up to weight 4 inclusive.

The expanded generating function is

$$1+x+y+2x^2+2xy+2y^2+3x^3+4x^2y+4xy^2+3y^3$$
$$+5x^4+7x^3y+9x^2y^2+7xy^3+5y^4+\ldots,$$

and we have

WEIGHT 1,

Biweight 10,	Biweight 01,
$(\overline{10})$;	$(\overline{01})$;

WEIGHT 2,

Biweight 20,	Biweight 11,	Biweight 02,
$(\overline{20})$	$(\overline{11})$	$(\overline{02})$
$(\overline{10^2})$;	$(\overline{10}\ \overline{01})$;	$(\overline{01^2})$;

WEIGHT 3,

Biweight 30,	Biweight 21,	Biweight 12,	Biweight 03,
$(\overline{30})$	$(\overline{21})$	$(\overline{12})$	$(\overline{03})$
$(\overline{20}\ \overline{10})$	$(\overline{20}\ \overline{01})$	$(\overline{10}\ \overline{02})$	$(\overline{01}\ \overline{02})$
$(\overline{10^3})$;	$(\overline{11}\ \overline{10})$	$(\overline{01}\ \overline{11})$	$(\overline{01^3})$;
	$(\overline{10^2}\ \overline{01})$;	$(\overline{10}\ \overline{01^2})$;	

WEIGHT 4,

Biweight 40,	Biweight 31,	Biweight 22,	Biweight 13,	Biweight 04,
$(\overline{40})$	$(\overline{31})$	$(\overline{22})$	$(\overline{13})$	$(\overline{04})$
$(\overline{30}\ \overline{10})$	$(\overline{21}\ \overline{10})$	$(\overline{21}\ \overline{01})$	$(\overline{12}\ \overline{01})$	$(\overline{03}\ \overline{01})$
$(\overline{20^2})$	$(\overline{30}\ \overline{01})$	$(\overline{12}\ \overline{10})$	$(\overline{03}\ \overline{10})$	$(\overline{02^2})$
$(\overline{20}\ \overline{10^2})$	$(\overline{20}\ \overline{11})$	$(\overline{20}\ \overline{02})$	$(\overline{02}\ \overline{11})$	$(\overline{02}\ \overline{01^2})$
$(\overline{10^4})$;	$(\overline{20}\ \overline{10}\ \overline{01})$	$(\overline{11^2})$	$(\overline{02}\ \overline{10}\ \overline{01})$	$(\overline{01^4})$.
	$(\overline{11}\ \overline{10^2})$	$(\overline{20}\ \overline{01^2})$	$(\overline{11}\ \overline{01^2})$	
	$(\overline{10^3}\ \overline{01})$;	$(\overline{02}\ \overline{10^2})$	$(\overline{10}\ \overline{01^3})$;	
		$(\overline{11}\ \overline{10}\ \overline{01})$		
		$(\overline{10^2}\ \overline{01^2})$;		

§ 2. *Preliminary Algebraic Theory.*

7. The partitions with one bipart correspond to the sums of the powers in the single system, or unipartite theory. They are easily expressed in terms of the fundamental symmetric functions.

The right hand side of the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots,$$

may be written

$$\overline{\exp}\,(a_{10}x + a_{01}y),$$

or, since it is convenient to write the symmetric function $(\overline{pq})$ in the form s_{pq}, this is

$$\overline{\exp}\,(s_{10}x + s_{01}y),$$

where the bar over exp indicates a symbolism by which $\dfrac{s_{10}{}^p s_{01}{}^q}{p!\, q!}$ denotes $(\overline{10}^p\,\overline{01}^q) \equiv a_{pq}$.

Hence the relation

$$1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots = \overline{\exp}\,(s_{10}x + s_{01}y),$$

which is important in connection with the collateral theory of operations to be presently brought into view.

8. Taking logarithms of both sides of the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots,$$

there results

$$s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2) + \tfrac{1}{3}(s_{30}x^3 + 3s_{21}x^2y + 3s_{12}xy^2 + s_{03}y^3) - \ldots *$$
$$= \log(1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots).$$

Hence

$$1 + a_{10}x + a_{01}y + \ldots a_{pq}x^p y^q + \ldots = \overline{\exp}\,(s_{10}x + s_{01}y)$$
$$= \exp\{s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2)$$
$$+ \tfrac{1}{3}(s_{30}x^3 + 3s_{21}x^2y + 3s_{12}xy^2 + s_{03}y^3) - \ldots\}.$$

Also we have the series of relations :—

$$\begin{cases} s_{10} = a_{10}, \\ s_{01} = a_{01}, \end{cases}$$
$$\begin{cases} s_{20} = a_{10}{}^2 - 2a_{20}, \\ s_{11} = a_{10}a_{01} - a_{11}, \\ s_{02} = a_{01}{}^2 - 2a_{02}, \end{cases}$$
$$\begin{cases} s_{30} = a_{10}{}^3 - 3a_{20}a_{10} + 3a_{30}, \\ s_{21} = a_{10}{}^2 a_{01} - a_{20}a_{01} - a_{11}a_{10} + a_{21}, \\ s_{12} = a_{01}{}^2 a_{10} - a_{02}a_{10} - a_{11}a_{01} + a_{12}, \\ s_{03} = a_{01}{}^3 - 3a_{02}a_{01} + 3a_{03}, \end{cases}$$
$$\vdots$$

* Viz :- $s_{10}x + s_{01}y = \Sigma(\alpha_1 x + \beta_1 y)$; $s_{20}x^2 + 2s_{11}xy + s_{02}y^2 = \Sigma(\alpha_1 x + \beta_1 y)^2$; &c.

and in general

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}s_{pq}=\Sigma_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}a_{p_1q_1}^{\pi_1}a_{p_2q_2}^{\pi_2}\ldots$$

9. Moreover, the fundamental symmetric functions are expressed in the terms of the forms s_{pq} by the formula

$$(-)^{p+q-1}a_{pq}=\Sigma\left\{\frac{(p_1+q_1-1)!}{p_1!\,q_1!}\right\}^{\pi_1}\left\{\frac{(p_2+q_2-1)!}{p_2!\,q_2!\ldots}\right\}^{\pi_2}\ldots\frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\ldots}s_{p_1q_1}^{\pi_1}s_{p_2q_2}^{\pi_2}\ldots,$$

as will be evident by simply applying the multinomial theorem to one of the above written general identities of Art. 8.

10. The single-bipart functions having been actually expressed in terms of the fundamental symmetric functions, it remains to show that all other rational algebraic symmetric functions are also so expressible. SCHLÄFLI (*loc. cit.*) has established this by induction, and it is not necessary to further discuss the theorem here. In Art. 43 of the present memoir, will be found the actual expression of a given symmetric function by means of single-bipart forms, a formula which, combined with one given above, Art. 8, serves to establish the theorem conclusively.

The Symmetric Function h_{pq}.

11. Write

$$(1+\alpha_1x+\beta_1y)(1+\alpha_2x+\beta_2y)\ldots=1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots$$
$$=\frac{1}{1-h_{10}x-h_{01}y+\ldots+(-)^{p+q}h_{pq}x^py^q+\ldots},$$

as the definition of the function h_{pq}.

Writing $-x$, $-y$ for x, y, we have

$$1+h_{10}x+h_{01}y+\ldots+h_{pq}x^py^q=\frac{1}{(1-\alpha_1x-\beta_1y)(1-\alpha_2x-\beta_2y)\ldots},$$

and expanding the right hand in ascending powers of x and y

$$h_{pq}=\Sigma\frac{(p_1+q_1)!}{p_1!\,q_1!}\frac{(p_2+q_2)!}{p_2!\,q_2!}\ldots(\overline{p_1q_1}\;\overline{p_2q_2}\ldots),$$

the summation being for all partitions of the biweight. Changing the signs of x and y in the relation first written down, we obtain

$$1+h_{10}x+h_{01}y+\ldots+h_{pq}x^py^q+\ldots=\frac{1}{1-a_{10}x-a_{01}y+\ldots+(-)^{p+q}a_{pq}x^py^q+\ldots},$$

an identity which arises from the former by interchanging the letter h with the letter a.

Hence, if f and ϕ be any two functions, such that

$$f(a_{10}, a_{01}, \ldots a_{pq}, \ldots) = \phi(h_{10}, h_{01}, \ldots h_{pq}, \ldots),$$

then also

$$\phi(a_{10}, a_{01}, \ldots a_{pq}, \ldots) = f(h_{10}, h_{01}, \ldots h_{pq}, \ldots);$$

and, in general, in any relation connecting the functions a with the functions h, an identity will still remain if the letters a and h be transposed.

By the multinomial theorem

$$(-)^{p+q-1} h_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi)!}{\pi_1!\,\pi_2!\ldots} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2} \ldots$$

From a previous result in this article, by taking logarithms and expanding

$$\frac{(p+q-1)!}{p!\,q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} h_{p_1q_1}^{\pi_1} h_{p_2q_2}^{\pi_2} \ldots,$$

which is to be compared with the formula

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2} \ldots,$$

and it will be noticed that s_{pq} remains unchanged when h is written for a, except for a change of sign, when the weight $p + q$ is even.

§ 3. *The Differential Operations.*

12. The beautiful properties of these symmetric functions are most easily established by means of the differential operations whose theory I proceed to establish.

Consider the identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_n x + \beta_n y)$$
$$= 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots,$$

where n may be as large as we please.

Multiply each side by $(1 + \mu x + \nu y)$.

The right hand side becomes

$$1 + (a_{10} + \mu) x + (a_{01} + \nu) y + (a_{20} + \mu a_{10}) x^2 + (a_{11} + \mu a_{01} + \nu a_{10}) xy$$
$$+ (a_{02} + \nu a_{01}) y^2 + \ldots,$$

and, in general, a_{pq} becomes converted into

$$a_{pq} + \mu a_{p-1,q} + \nu a_{p,q-1}.$$

Hence any rational integral function of the coefficients

$$a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \ldots,$$

viz.,

$$f(a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \ldots) \equiv f,$$

is converted into

$$f + (\mu g_{10} + \nu g_{01}) f + \frac{1}{2!}(\mu g_{10} + \nu g_{01})^2 f + \frac{1}{3!}(\mu g_{10} + \nu g_{01})^3 f + \ldots,$$

where

$$g_{10} = \Sigma a_{p-1,q}\, \partial_{a_{pq}}; \quad g_{01} = \Sigma a_{p,q-1}\, \partial_{a_{pq}};$$

and the multiplication of operators is symbolic.*

The new value of f is

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01}) f,$$

where the bar is placed over exp to denote that the multiplication of operators is symbolic (*vide* Art. 7).

13. Write

$$G_{pq} = \frac{1}{p!\,q!}\overline{g_{10}{}^p g_{01}{}^q},$$

the bar denoting symbolic multiplication, then

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01}) f$$
$$= (1 + \mu G_{10} + \nu G_{01} + \mu^2 G_{20} + \mu\nu G_{11} + \nu^2 G_{02} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f.$$

(Compare Art. 7.)

Now suppose the symmetric function f expressed in terms of

$$\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \ldots \alpha_n, \beta_n$$

to be

$$(\overline{p_1 q_1}\; \overline{p_2 q_2}\; \overline{p_3 q_3} \ldots).$$

The introduction of the new quantities μ, ν results in the addition to

$$(\overline{p_1 q_1}\; \overline{p_2 q_2}\; \overline{p_3 q_3} \ldots)$$

of the terms

$$\mu^{p_1}\nu^{q_1}(\overline{p_2 q_2}\; \overline{p_3 q_3} \ldots) + \mu^{p_2}\nu^{q_2}(\overline{p_1 q_1}\; \overline{p_3 q_3} \ldots) + \mu^{p_3}\nu^{q_3}(\overline{p_1 q_1}\; \overline{p_2 q_2} \ldots) + \ldots;$$

* By "symbolic" is to be understood "non-operational," as in what is commonly known as the "symbolic" form of TAYLOR'S Theorem.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and hence

$$f + \mu^{p_1}\nu^{q_1}(\overline{p_2q_2}\,\overline{p_3q_3}\ldots) + \mu^{p_2}\nu^{q_2}(\overline{p_1q_1}\,\overline{p_3q_3}\ldots) + \mu^{p_3}\nu^{q_3}(\overline{p_1q_1}\,\overline{p_2q_2}\ldots) + \ldots$$
$$= (1 + \mu G_{10} + \nu G_{01} + \mu^2 G_{20} + \mu\nu G_{11} + \nu^2 G_{02} + \ldots) f;$$

and equating coefficients of like products $\mu^p\nu^q$, we find

$$G_{p_1q_1}(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\ldots) = (\overline{p_2q_2}\,\overline{p_3q_3}\ldots),$$
$$G_{p_2q_2}(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\ldots) = (\overline{p_1q_1}\,\overline{p_3q_3}\ldots),$$
$$G_{p_3q_3}(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\ldots) = (\overline{p_1q_1}\,\overline{p_2q_2}\ldots),$$
$$G_{p_1q_1}(\overline{p_1q_1}) = 1,$$
$$G_{p_1q_1}G_{p_2q_2}\ldots G_{p_nq_n}(\overline{p_1q_1}\,\overline{p_2q_2}\ldots\overline{p_nq_n}) = 1,$$

and $G_{rs}f = 0$, unless the bipart $\overline{rs}$ is involved in the expression of f.

From the above we gather the very important fact that the effect of the operation G_{pq} upon a partition is to obliterate one bipart $\overline{pq}$ when such bipart is present, and to annihilate the partition if it contains no bipart $\overline{pq}$.

14. I return to the result

$$1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots = \overline{\exp}\,(\mu g_{10} + \nu g_{01}),$$

wherein be it remembered the multiplication of operators in the right hand expression is symbolic. I seek to replace $\overline{\exp}\,(\mu g_{10} + \nu g_{01})$ by an expression containing products of linear partial differential operations in which the multiplication is not symbolic.

We have by definition

$$g_{10} = \partial_{a_{10}} + a_{10}\,\partial_{a_{20}} + a_{01}\,\partial_{a_{11}} + \ldots;$$
$$g_{01} = \partial_{a_{01}} + a_{01}\,\partial_{a_{02}} + a_{10}\,\partial_{a_{11}} + \ldots;$$

let further

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,q}} + a_{01}\,\partial_{a_{p,q+1}} + \ldots + a_{rs}\,\partial_{a_{p+r,q+s}} + \ldots,$$

a definition which includes the former.

15. I will establish the relation

$$\overline{\exp}\,(m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots)$$
$$= \exp\,(M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots),$$

where on the left and right hand sides the multiplications of operators are respectively symbolic and not symbolic, and

$$\exp (M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots)$$
$$= 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots,$$

where ξ, η are the undetermined algebraic quantities.

For the multiplication of two operators $g_{p_1q_1}$, $g_{p_2q_2}$, we have the formula

$$g_{p_1q_1}\, g_{p_2q_2} = \overline{g_{p_1q_1}\, g_{p_2q_2}} + g_{p_1q_1} \dagger g_{p_2q_2},$$

wherein the symbol $\dagger$ denotes explicit operation upon the operand, regarding the latter as a function of symbols of quantity only, and not of the differential inverses.

Also

$$g_{p_1q_1} \dagger g_{p_2q_2} = g_{p_1+p_2,\, q_1+q_2}.$$

Put

$$u_1 = m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots,$$

which may be written

$$u_1 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)\, g,$$

in which $m_{pq}g_{pq}$ is symbolically written $m_{pq}g$.

But

$$u_2 = u_1 \dagger u_1 = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^2 g,$$

where, after expansion of the right hand side, $m_{p_1q_1}m_{p_2q_2}g$ is to be written

$$m_{p_1q_1}m_{p_2q_2}g_{p_1+p_2,\ q_1+q_2}.$$

Then, with a similar convention,

$$u_s = u_1 \dagger u_{s-1} = (m_{10} + m_{01} + \ldots + m_{pq} + \ldots)^s g.$$

Further, it is easy to prove the relation

$$u_s \dagger u_t = u_{s+t}.$$

But for a series of linear partial differential operators enjoying this property, it is a well known and easily established theorem of SYLVESTER'S that

$$\overline{\exp}\, u_1 = \exp (u_1 - \tfrac{1}{2}\, u_2 + \tfrac{1}{3}\, u_3 - \ldots).$$

Hence, substituting, we easily reach the relation

$$\exp (M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots) = 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots,$$

wherein ξ and η are undetermined algebraic quantities.

This establishes the theorem.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

16. To apply it to the case in hand, put

$$m_{10} = \mu, \qquad m_{01} = \nu, \qquad m_{pq} = 0 \text{ in other cases.}$$

Then

$$M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots = \log(1 + \mu\xi + \nu\eta)$$

$$\therefore \quad M_{pq} = (-)^{p+q+1}\frac{(p+q-1)!}{p!\,q!}\mu^p\nu^q,$$

and the result is

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01}) = \exp\{\mu g_{10} + \nu g_{01} - \tfrac{1}{2}(\mu^2 g_{20} + 2\mu\nu g_{11} + \nu^2 g_{02}) + \tfrac{1}{3}(\mu^3 g_{30} + 3\mu^2\nu g_{21} + 3\mu\nu^2 g_{12} + \nu^3 g_{03}) - \ldots\}.$$

Combining with a former result

$$\mu g_{10} + \nu g_{01} - \tfrac{1}{2}(\mu^2 g_{20} + 2\mu\nu g_{11} + \nu^2 g_{02}) + \tfrac{1}{3}(\mu^3 g_{30} + 3\mu^2\nu g_{21} + 3\mu\nu^2 g_{12}\,\nu^3 g_{03}) - \ldots = \log(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots).$$

(Compare Art. 8.)

17. By expanding the right hand side we can express each linear operator of the form g_{pq} in terms of products of the obliterating operators which have the form G_{pq}.

The law is identical with that which expresses the functions containing one bipart in terms of the fundamental symmetric functions. We find

$$\begin{cases} g_{10} = G_{10} \\ g_{01} = G_{01} \end{cases}$$

$$\begin{cases} g_{20} = G_{10}^2 - 2G_{20} \\ g_{11} = G_{10}G_{01} - G_{11} \\ g_{02} = G_{01}^2 - 2G_{02} \end{cases}$$

$$\begin{cases} g_{30} = G_{10}^3 - 3G_{20}G_{10} + 3G_{30} \\ g_{21} = G_{10}^2 G_{01} - G_{20}G_{01} - G_{11}G_{10} + G_{21} \\ g_{12} = G_{01}^2 G_{10} - G_{02}G_{10} - G_{11}G_{01} + G_{12} \\ g_{03} = G_{01}^3 - 3G_{02}G_{01} + 3G_{03} \end{cases}$$

$$\vdots$$

and in general

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma_\pi(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}G_{p_1q_1}^{\pi_1}\;G_{p_2q_2}^{\pi_2}\;\ldots$$

while

$$(-)^{p+q-1}G_{pq} = \Sigma_\pi\left\{\frac{(p_1+q_1-1)!}{p_1!\,q_1!}\right\}^{\pi_1}\left\{\frac{(p_2+q_2-1)!}{p_2!\,q_2!}\right\}^{\pi_2}\ldots\frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\ldots}g_{p_1q_1}^{\pi_1}\;g_{p_2q_2}^{\pi_2}\;\ldots$$

(Compare Art. 8.)

18. By comparison of these relations with the corresponding algebraic ones to which reference has been made, it is manifest that g_{pq} and G_{pq} are respectively in co-relation with s_{pq} and a_{pq}. In other words these operations respectively correspond to the partitions $(\overline{pq})$ and $(\overline{10}^p\ \overline{01}^q)$. It is necessary to find the operations which correspond to the remaining partitions which symbolize symmetric functions.

We have the easily derivable results in operations

$$g_{p_1q_1}g_{p_2q_2} = \overline{g_{p_1q_1}g_{p_2q_2}} + g_{p_1+p_2,\, q_1+q_2},$$
$$g_{p_1q_2}{}^2 = \overline{g_{p_1q_2}{}^2} + g_{2p_1,\, 2q_1},$$
$$g_{p_1q_1}g_{p_2q_2}g_{p_3q_3} = \overline{g_{p_1q_1}g_{p_2q_2}g_{p_3q_3}} + \overline{g_{p_1q_1}g_{p_2+p_3,\, q_2+q_3}} + \overline{g_{p_2q_2}g_{p_3+p_1,\, q_3+q_1}}$$
$$+ \overline{g_{p_3q_3}g_{p_1+p_2,\, q_1+q_2}} + g_{p_1+p_2+p_3,\, q_1+q_2+q_3},$$
$$g_{p_1q_1}{}^2g_{p_2q_2} = \overline{g_{p_1q_1}{}^2g_{p_2q_2}} + 2\overline{g_{p_1q_1}g_{p_1+p_2,\, q_1+q_2}} + \overline{g_{2p_1,\, 2q_1}g_{p_2q_2}} + g_{2p_1+p_2,\, 2q_1+q_2},$$
$$g_{p_1q_1}{}^3 = \overline{g_{p_1q_1}{}^3} + 3\overline{g_{2p_1,\, 2q_1}g_{p_1q_1}} + g_{3p_1,\, 3q_1};$$

where as usual the bar denotes symbolic multiplication; and comparing these with the algebraic formulæ

$$(\overline{p_1q_1})(\overline{p_2q_2}) = (\overline{p_1q_1}\ \overline{p_2q_2}) + (\overline{p_1+p_2,\, q_1+q_2})$$
$$(\overline{p_1q_1})^2 = 2(\overline{p_1q_1}^2) + (\overline{2p_1,\, 2q_1})$$
$$(\overline{p_1q_1})(\overline{p_2q_2})(\overline{p_3q_3}) = (\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3}) + (\overline{p_1q_1}\ \overline{p_2+p_3,\, q_2+q_3}) + (\overline{p_2q_2}\ \overline{p_3+p_1,\, q_3+q_1})$$
$$+ (\overline{p_3q_3}\ \overline{p_1+p_2,\, q_1+q_2}) + (\overline{p_1+p_2+p_3,\, q_1+q_2+q_3})$$
$$(\overline{p_1q_1})^2(\overline{p_2q_2}) = 2(\overline{p_1q_1}^2\ \overline{p_2q_2}) + 2(\overline{p_1q_1}\ \overline{p_1+p_2,\, q_1+q_2}) + (\overline{2p_1,\, 2q_1}\ \overline{p_2q_2})$$
$$+ (\overline{2p_1+p_2,\, 2q_1+q_2})$$
$$(\overline{p_1q_1})^3 = 6(\overline{p_1q_1}^3) + 3(\overline{2p_1,\, 2q_1}\ \overline{p_1q_1}) + (\overline{3p_1,\, 3q_1})$$

it is evident that the operations

$$\frac{\overline{g_{p_1q_1}{}^2}}{2\,!}, \quad \frac{\overline{g_{p_1q_1}{}^2g_{p_2q_2}}}{2\,!}, \quad \frac{\overline{g_{p_1q_1}{}^3}}{3\,!},$$

are produced according to the same *law* as the symmetric functions

$$(\overline{p_1q_1}^2), \quad (\overline{p_1q_1}^2\,\overline{p_2q_2}), \quad (\overline{p_1q_1}^3);$$

further the *law* is perfectly general and indicates that the operation

$$\frac{1}{\pi_1\,!}\,\frac{1}{\pi_2\,!}\cdots\overline{g_{p_1q_1}{}^{\pi_1}g_{p_2q_2}{}^{\pi_2}\cdots},$$

is in co-relation with the symmetric function

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots).$$

19. There is thus complete correspondence between quantity and operation, and any formula of quantity may be at once translated into a formula of operation.

Observe that a product of symmetric functions

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)\,(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$$

is in correspondence with the operation

$$\frac{1}{\pi_1!}\,\frac{1}{\pi_2!}\ldots\overline{g_{p_1q_1}^{\ \pi_1}\,g_{p_2q_2}^{\ \pi_2}\ldots}\;\frac{1}{\rho_1!}\,\frac{1}{\rho_2!}\ldots\overline{g_{r_1s_1}^{\ \rho_1}\,g_{r_2s_2}^{\ \rho_2}\ldots}$$

the notation indicating that the two operations

$$\frac{1}{\rho_1!}\,\frac{1}{\rho_2!}\ldots\overline{g_{r_1s_1}^{\ \rho_1}\,g_{r_2s_2}^{\ \rho_2}\ldots}\quad\text{and}\quad\frac{1}{\pi_1!}\,\frac{1}{\pi_2!}\ldots\overline{g_{p_1q_1}^{\ \pi_1}\,g_{p_2q_2}^{\ \pi_2}\ldots}$$

are to be successively performed.

For an example take the algebraic formula

$$(\overline{31}\,\overline{01})=-\tfrac{1}{2}(\overline{21}\,\overline{10})(\overline{01})+\tfrac{1}{2}(\overline{21}\,\overline{01})(\overline{10})+\tfrac{1}{2}(\overline{10}\,\overline{01})(\overline{21})-\tfrac{1}{2}(\overline{21}\,\overline{10}\,\overline{01}),$$

which is translated into the operator formula

$$\overline{g_{31}g_{01}}=-\tfrac{1}{2}\overline{g_{21}g_{10}}\cdot g_{01}+\tfrac{1}{2}\overline{g_{21}g_{01}}\cdot g_{10}+\tfrac{1}{2}\overline{g_{10}g_{01}}\cdot g_{21}-\tfrac{1}{2}\overline{g_{21}g_{10}g_{01}}.$$

20. It is now necessary to enquire into the laws which appertain to the performance of these operations upon symmetric functions. We have seen, *ante* Art. 13, the law by which the obliterator G_{pq} is performed upon a monomial symmetric function.

Since

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq}=\Sigma_{\pi}(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!}\,G_{p_1q_1}^{\pi_1}\;G_{p_2q_2}^{\pi_2}\;\ldots,$$

we can operate with g_{pq} upon a monomial form by operating independently with the successive G products on the right, and adding the results together. As a particular result, observe that a term on the right is G_{pq}, and hence

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq}s_{pq}=G_{pq}s_{pq}=G_{pq}(\overline{pq})=1,$$

or

$$g_{pq}s_{pq}=(-)^{p+q-1}\frac{p!\,q!}{(p+q-1)!};$$

and g_{pq} causes every other single bipart function to vanish; it must, indeed, cause any monomial function to vanish which does not comprise one of the partitions of the biweight pq amongst its biparts.

21. The relation just obtained yields the equivalence

$$g_{pq} = (-)^{p+q-1} \frac{p!\,q!}{(p+q-1)!} \partial_{s_{pq}},$$

and further results of the nature

$$\overline{g_{p_1q_1} g_{p_2q_2}} = (-)^{p_1+p_2+q_1+q_2} \left\{ \frac{p_1!\,q_1!\,p_2!\,q_2!}{(p_1+q_1-1)!\,(p_2+q_2-1)!} \partial_{s_{p_1q_1}} \partial_{s_{p_2q_2}} + \frac{(p_1+p_2)!\,(q_1+q_2)!}{(p_1+p_2+q_1+q_2-1)!} \partial_{s_{p_1+p_2,\,q_1+q_2}} \right\},$$

which are of use in connexion with the theory of function with single biparts.

Since every symmetric function is expressible in terms of the fundamental symmetric functions, every operation $\overline{g_{p_1q_1}^{\pi_1} g_{p_2q_2}^{\pi_2}}$ is necessarily expressible as a sum of G products and can be performed upon a monomial symmetric function.

22. The solutions of the partial differential equation

$$g_{pq} = 0,$$

are the single bipart forms omitting s_{pq} (Art. 21), while the solution of the partial differential equation

$$G_{pq} = 0,$$

are those monomial symmetric functions in which the bipart $\overline{pq}$ is absent (Art. 13).

23. The operation $\partial_{a_{pq}}$ is expressible by means of the operations g_{pq}.

Reversing the formula

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,q}} + a_{01}\,\partial_{a_{p,q+1}} + \ldots + a_{rs}\,\partial_{a_{p+r,q+s}} + \ldots,$$

we obtain

$$\partial_{a_{pq}} = g_{pq} - h_{10} g_{p+1,q} - h_{01} g_{p,q+1} + \ldots + (-)^{r+s} h_{rs} g_{p+r,q+s} + \ldots,$$

where as before (Art. 11),

$$(-)^{r+s-1} h_{rs} = \Sigma_\rho\, (-)^{\Sigma\rho-1} \frac{(\Sigma\rho)!}{\rho_1!\,\rho_2!} a_{r_1s_1}^{\rho_1}\, a_{r_2s_2}^{\rho_2} \ldots .$$

§ 4. *The Theory of Three Identities.*

24. The course of the investigation at this point necessitates the introduction of two identities similar to, and in addition to, the fundamental identity.

Let

$$1+a_{10}x+a_{01}y+\ldots+a_{pq}x^py^q+\ldots=(1+\alpha_1x+\beta_1y)(1+\alpha_2x+\beta_2y)\ldots \quad \text{(I.)}$$

$$1+b_{10}x+b_{01}y+\ldots+b_{pq}x^py^q+\ldots=(1+\alpha_1^{(1)}x+\beta_1^{(1)}y)(1+\alpha_2^{(1)}x+\beta_2^{(1)}y)\ldots \quad \text{(II.)}$$

$$1+c_{10}x+c_{01}y+\ldots+c_{pq}x^py^q+\ldots=(1+\alpha_1^{(2)}x+\beta_1^{(2)}y)(1+\alpha_2^{(2)}x+\beta_2^{(2)}y)\ldots \quad \text{(III.)}$$

wherein x and y may be regarded as undetermined quantities and the identities as merely expressing the relations between the coefficients on the left and quantities α, β on the right.

Assume the coefficients and quantities in the first two identities to be given and the coefficients in the third identity to be then determined by the relations:—

$$\begin{aligned}1+c_{01}\xi+c_{01}\eta+\ldots+c_{pq}\xi^p\eta^q+\ldots \\ =\Pi_s(1+\alpha_sb_{10}\xi+\beta_sb_{01}\eta+\ldots+\alpha_s^p\beta_s^qb_{pq}\xi^p\eta^q+\ldots),\end{aligned}$$

ξ and η being undetermined quantities.

Multiplying out the right-hand side of this relation, it is found to be equivalent to the series of relations:—

$$\begin{aligned}
c_{10}&=(\overline{10})\,b_{10}\,,\\
c_{01}&=(\overline{01})\,b_{01}\,,\\
c_{20}&=(\overline{20})\,b_{20}+(\overline{10}^2)\,b_{10}^2\,,\\
c_{11}&=(\overline{11})\,b_{11}+(\overline{10}\,\overline{01})\,b_{10}b_{01}\,,\\
c_{02}&=(\overline{02})\,b_{02}+(\overline{01}^2)\,b_{01}^2\,,\\
c_{30}&=(\overline{30})\,b_{30}+(\overline{20}\,\overline{10})\,b_{20}b_{10}+(\overline{10}^3)\,b_{10}^3\,,\\
c_{21}&=(\overline{21})\,b_{21}+(\overline{20}\,\overline{01})\,b_{20}b_{01}+(\overline{11}\,\overline{10})\,b_{11}b_{10}+(\overline{10}^2\,\overline{01})\,b_{10}^2b_{01}\,,\\
c_{12}&=(\overline{12})\,b_{12}+(\overline{02}\,\overline{10})\,b_{02}b_{10}+(\overline{11}\,\overline{01})\,b_{11}b_{01}+(\overline{10}\,\overline{01}^2)\,b_{10}b_{01}^2\,,\\
c_{03}&=(\overline{03})\,b_{03}+(\overline{02}\,\overline{01})\,b_{02}b_{01}+(\overline{01}^3)\,b_{01}^3\,;\\
&\vdots\quad\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and generally in the expression of c_{pq} every symmetric function of biweight pq of the quantities in the first identity occurs, each attached to the corresponding product of coefficients from the second identity.

25. Represent the symmetric functions of the quantities occurring in the second and third identities by partitions in brackets $(\quad)_1$, $(\quad)_2$, respectively.

Now

$$\Pi_s(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p \beta_s^q b_{pq}\xi^p\eta^q + \ldots)$$

is from the identity II. equal to

$$\Pi_s[(1 + \alpha_s\alpha_1^{(1)}\xi + \beta_s\beta_1^{(1)}\eta)(1 + \alpha_s\alpha_2^{(1)}\xi + \beta_s\beta_2^{(1)}\eta)(\ldots)\ldots],$$

which is

$$\Pi_s\Pi_t(1 + \alpha_s\alpha_t^{(1)}\xi + \beta_s\beta_t^{(1)}\eta).$$

26. Hence the assumed relation becomes on taking logarithms

$$\Sigma_s \log(1 + \alpha_s^{(2)}\xi + \beta_s^{(2)}\eta) = \Sigma_s\Sigma_t \log(1 + \alpha_s\alpha_t^{(1)}\xi + \beta_s\beta_t^{(1)}\eta),$$

and expanding and equating coefficients of $\xi^p\eta^q$

$$(\overline{pq})_2 = (\overline{pq})(\overline{pq})_1;$$

an important relation which shows that the assumed relation is unaltered when the set of quantities α is interchanged with the set $\alpha^{(1)}$, in such wise that α_s and $\alpha_s^{(1)}$ are transposed. It is indeed of fundamental importance, and will be brought prominently forward in the sequel. Its consideration must be postponed until a further step has been taken in the theory of the operators.

27. Let the operators

$$\begin{array}{ll} g_{pq}, & G_{pq}, \\ g_{pq}', & G_{pq}', \\ g_{pq}'', & G_{pq}'', \end{array}$$

refer to identities I., II., III., respectively.

Writing the relation

$$\begin{aligned} 1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots \\ = \Pi_s(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p\beta_s^q b_{pq}\xi^p\eta^q + \ldots), \end{aligned}$$

in the abbreviated form

$$U = u_{\alpha_1\beta_1}u_{\alpha_2\beta_2}u_{\alpha_3\beta_3}\ldots,$$

and performing the operation

$$g_{pq}' = \partial_{b_{pq}} + b_{10}\,\partial_{b_{p+1,q}} + b_{01}\,\partial_{b_{p,q+1}} + \ldots + b_{rs}\,\partial_{b_{p+r,q+s}} + \ldots$$

we have

$$g_{pq}'\mathrm{U} = (g_{pq}'u_{\alpha_1\beta_1})\,u_{\alpha_2\beta_2}u_{\alpha_3\beta_3}\ldots + u_{\alpha_1\beta_1}(g_{pq}'u_{\alpha_2\beta_2})\,u_{\alpha_3\beta_3} + \ldots$$

Moreover,

$$g_{pq}'u_{\alpha_s\beta_s} = \alpha_s^p\beta_s^q\xi^p\eta^q u_{\alpha_s\beta_s};$$

hence

$$g_{pq}'\mathrm{U} = (\overline{pq})\xi^p\eta^q\mathrm{U},$$

and replacing U by its value, we have

$$g_{pq}'c_{pq} = (\overline{pq});$$

while in general

$$g_{pq}'c_{rs} = (\overline{pq})\,c_{r-p,s-q}.$$

Now regarding the coefficients b_{pq} as functions of the coefficients c_{pq} only, we have

$$g_{pq}' = (g_{pq}'c_{pq})\,\partial_{c_{pq}} + \ldots + (g_{pq}'c_{rs})\,\partial_{c_{rs}} + \ldots$$
$$= (\overline{pq})\,(\partial_{c_{pq}} + c_{10}\,\partial_{c_{p+1,q}} + c_{01}\,\partial_{c_{p,q+1}} + \ldots + c_{r-p,s-q}\,\partial_{c_{rs}} + \ldots)$$

Thus

$$g_{pq}' = (\overline{pq})\,g_{pq}''.$$

But the assumed relation is symmetrical as regards the quantities in the first two identities; hence also

$$g_{pq} = (\overline{pq})_1\,g_{pq}'',$$

and thence, since $(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1$, we have

$$(\overline{pq})_2\,g_{pq}'' = (\overline{pq})_1\,g_{pq}' = (\overline{pq})\,g_{pq}.$$

If we then regard the assumed relation as defining a transformation of the quantities occurring in the identity III. into either of the sets of quantities associated with I. or II., the operation

$$(\overline{pq})_2\,g_{pq}''$$

is an invariant.

28. Since $g_{pq}' = (\overline{pq})\,g_{pq}''$

$$\xi g_{10}' + \eta g_{01}' - \tfrac{1}{2}(\xi^2 g_{20}' + 2\xi\eta g_{11}' + \eta^2 g_{02}') + \tfrac{1}{3}(\xi^3 g_{30}' + 3\xi^2\eta g_{21}' + 3\xi\eta^2 g_{12}' + \eta^3 g_{03}') - \ldots$$
$$= \xi(\overline{10})\,g_{10}'' + \eta(\overline{01})\,g_{01}'' - \tfrac{1}{2}\{\xi^2(\overline{20})\,g_{20}'' + 2\xi\eta(\overline{11})\,g_{11}'' + \eta^2(\overline{02})\,g_{02}''\}$$
$$+ \tfrac{1}{3}\{\xi^3(\overline{30})\,g_{30}'' + 3\xi^2\eta(\overline{21})\,g_{21}'' + 3\xi\eta^2(\overline{12})\,g_{12}'' + \eta^3(\overline{03})\,g_{03}''\} - \ldots$$

The left hand side of this equation is

$$\log\,(1 + \xi G_{10}' + \eta G_{01}' + \ldots + \xi^p \eta^q G_{pq}' + \ldots)\ (\textit{vide}\ \text{Art. } 16),$$

while if the operators g'' be replaced by their expressions in terms of the operators G'', it is easily seen that the right side is

$$\Sigma_s \log\,(1 + \alpha_s G_{10}''\xi + \beta_s G_{01}''\eta + \ldots + \alpha_s^p \beta_s^q G_{pq}'' \xi^p \eta^q + \ldots).$$

29. Hence the operator relation

$$\begin{aligned} &1 + G_{10}'\xi + G_{01}'\eta + \ldots + G_{pq}'\xi^p\eta^q + \ldots \\ &\quad = \Pi_s\,(1 + \alpha_s G_{10}''\xi + \beta_s G_{01}''\eta + \ldots + \alpha_s^p \beta_s^q G_{pq}''\xi^p\eta^q + \ldots). \end{aligned}$$

This result must be compared with the relation (Art. 24)

$$\begin{aligned} &1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots \\ &\quad = \Pi_s\,(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p \beta_s^q b_{pq}\xi^p\eta^q + \ldots). \end{aligned}$$

30. I say that such a comparison yields the following theorem :—

"In any relation connecting the quantities c_{pq} with the quantities b_{pq}, we are at liberty to substitute

$$G_{pq}' \text{ for } c_{pq}, \quad \text{and} \quad G_{pq}'' \text{ for } b_{pq};$$

and we in this manner obtain a relation between operators in correspondence."

To explain this further, observe that ξ, η being undetermined quantities in the assumed relation which connects the quantities of the three identities I., II., III., we are able to express any product whatever of the coefficients c_{10}, c_{01}, ... c_{pq}, ... in terms of products of coefficients b_{10}, b_{01}, ... b_{pq}, ... and of symmetrical functions of the quantities α_1, β_1, α_2, β_2, ... The substitution in question can be made in any equation thus formed.

31. With regard to the relation of Art. 24, viz. :—

$$\begin{aligned} &1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots \\ &\quad = \Pi_s\,(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p \beta_s^q b_{pq}\xi^p\eta^q + \ldots), \end{aligned}$$

two important facts have been established—

(i.) That the relation is unaltered when the quantities occurring in the first identity

$$1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^py^q + \ldots = (1 + \alpha_1 x + \beta_1 y)\,(1 + \alpha_2 x + \beta_2 y) \ldots$$

are exchanged with those occurring in the second identity

$$1 + b_{10}x + b_{01}y + \ldots + b_{pq}x^p y^q + \ldots = (1 + \alpha_1^{(1)}x + \beta_1^{(1)}y)(1 + \alpha_2^{(1)}x + \beta_2^{(1)}y)\ldots$$

each with each. (Art. 26.)

(ii.) That we can always proceed to a relation between the operations by writing

$$G_{pq}' \text{ for } c_{pq} \quad \text{and} \quad G_{pq}'' \text{ for } b_{pq}. \quad \text{(Art. 30.)}$$

I will refer to these facts as the first and second properties of the relation respectively.

§ 5. *The First Law of Symmetry.*

32. By means of the equality

$$(\overline{pq})_2 = (\overline{pq})(\overline{pq})_1$$

which has been established *ante* (Art. 26), it is clear that any symmetric function expressed in a bracket $(\quad)_2$ can be expressed as a linear function of products of symmetric functions of the form $(\quad)(\quad)_1$; it is also clear from the first property above defined, that such expression will remain unaltered when the brackets $(\quad)$ and $(\quad)_1$ are interchanged; it must, therefore, be a symmetric function in regard to these brackets.

We may, therefore, suppose an equation

$$(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)_2 = \ldots + \text{J}\,(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_1$$
$$+ \text{J}\,(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)_1(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots) + \ldots \qquad \text{(A)}$$

Moreover, we can express any product of the coefficients $c_{10}, c_{01}, \ldots c_{pq}, \ldots$ as a linear function of expressions each of which contains a monomial symmetric function of the quantities $\alpha_1, \beta_1; \alpha_2, \beta_2; \ldots$ and a product of coefficients $b_{10}, b_{01}, \ldots b_{pq}, \ldots$

Assume then

$$c_{p_1q_1}^{\pi_1}\,c_{p_2q_2}^{\pi_2}\ldots = \ldots + \text{L}\,(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)\,b_{r_1s_1}^{\rho_1}\,b_{r_2s_2}^{\rho_2}\ldots + \ldots, \qquad \text{(B)}$$

$$c_{a_1b_1}^{\alpha_1}\,c_{a_2b_2}^{\alpha_2}\ldots = \ldots + \text{M}\,(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)\,b_{r_1s_1}^{\rho_1}\,b_{r_2s_2}^{\rho_2}\ldots + \ldots, \qquad \text{(C)}$$

From equation (B) is derived by the second property the operator relation

$$G'^{\pi_1}_{p_1q_1}\,G'^{\pi_2}_{p_2q_2}\ldots = \ldots + \text{L}\,(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)\,G''^{\rho_1}_{r_1s_1}\,G''^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

and performing each side of this equation upon the opposite side of the equation (A) we obtain, after cancelling of $(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)$,

$$\mathrm{L}\mathrm{G''}^{\rho_1}_{r_1s_1}\,\mathrm{G''}^{\rho_2}_{r_2s_2}\ldots(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}\ldots)_2 = \mathrm{J}\mathrm{G'}^{\pi_1}_{p_1q_1}\,\mathrm{G'}^{\pi_2}_{p_2q_2}\ldots(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_1,$$

no other terms surviving the operations, or

$$\mathrm{L} = \mathrm{J}\,;$$

since the symmetric function on either side is reduced to unity by the operation.

Similarly the equation (C) yields the equation of operators

$$\mathrm{G'}^{\alpha_1}_{a_1b_1}\,\mathrm{G'}^{\alpha_2}_{a_2b_2}\ldots = \ldots + \mathrm{M}\,(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)\,\mathrm{G''}^{\rho_1}_{r_1s_1}\,\mathrm{G''}^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

and this when performed on opposite sides of equation (A) gives

$$\mathrm{M} = \mathrm{J}.$$

Hence

$$\mathrm{L} = \mathrm{M},$$

and we have the law of symmetry expressed by the two relations

$$c^{\pi_1}_{p_1q_1}\,c^{\pi_2}_{p_2q_2}\ldots = \ldots + \mathrm{L}\,(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)\,b^{\rho_1}_{r_1s_1}\,b^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

$$c^{\alpha_1}_{a_1b_1}\,c^{\alpha_2}_{a_2b_2}\ldots = \ldots + \mathrm{L}\,(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)\,b^{\rho_1}_{r_1s_1}\,b^{\rho_2}_{r_2s_2}\ldots + \ldots,$$

viz., if in the first of these relations the partitions $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, $(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)$ be interchanged the numerical coefficient L remains unaltered.

The theorem is a consequence of the two properties that have been established in regard to the three identities and the relation assumed to exist between the quantities involved in them.

It appears to be the most important theorem in symmetrical algebra.

33. I now pass to certain consequences which flow straight from the theorem.

It is necessary to make a few definitions.

Definition.

"A partition is separated into separates by writing down a set of partitions, each separate partition in its own brackets, so that when all the parts of these partitions are assembled in a single bracket, the partition which is separated is reproduced."

For the purpose of this portion of the memoir alone it would have been expedient to use the word bipart in lieu of the word part in the foregoing definition, but I have retained the word part for the reason that the definition remains valid whatever be the order of multiplicity of the parts.

Of a partition $(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3})$ the product $(\overline{p_1q_1}\ \overline{p_3q_3})\ (\overline{p_2q_2})$ is a separation composed of the separates $(\overline{p_1q_1}\ \overline{p_3q_3})$ and $(\overline{p_2q_2})$.

Definition.

"A partition *quâ* its separations is termed a separable partition."

Definition.

"If the successive biweights of the separates of a separation be

$$w_1^{(1)}w_2^{(1)},\quad w_1^{(2)}w_2^{(2)},\quad w_1^{(3)}w_2^{(3)}\ \ldots,$$

the separation is said to have the *specification*

$$(\overline{w_1^{(1)}w_2^{(1)}}\ \overline{w_1^{(2)}w_2^{(2)}}\ \overline{w_1^{(3)}w_2^{(3)}}\ \ldots)."$$

Observe that the biweights of the separation and of its specification are necessarily the same, and identical with the biweight of the separable partition.

Observation.

The separable partition is counted as one amongst its own separations.

34. To take a concrete example of these definitions, consider a separable partition $(\overline{20}\ \overline{10}\ \overline{01})$.

We have

Separations.	Specifications.
$(\overline{20}\ \overline{10}\ \overline{01})$	$(\overline{31})$,
$(\overline{20}\ \overline{10})\ (\overline{01})$	$(\overline{30}\ \overline{01})$,
$(\overline{20}\ \overline{01})\ (\overline{10})$	$(\overline{21}\ \overline{10})$,
$(\overline{10}\ \overline{01})\ (\overline{20})$	$(\overline{11}\ \overline{20})$,
$(\overline{20})\ (\overline{10})\ (\overline{01})$	$(\overline{20}\ \overline{10}\ \overline{01})$.

35. I will discuss the law of symmetry that has been established in the light of these definitions.

I recall the relations

$$c_{10} = (\overline{10})\, b_{10},$$
$$c_{01} = (\overline{01})\, b_{01},$$
$$c_{20} = (\overline{20})\, b_{20} + (\overline{10}^2)\, b_{10}^2,$$
$$c_{11} = (\overline{11})\, b_{11} + (\overline{10}\;\overline{01})\, b_{10} b_{01},$$
$$c_{02} = (\overline{02})\, b_{02} + (\overline{01}^2)\, b_{01}^2,$$
$$\vdots \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

In the expression of c_{pq} each partition in brackets ( ) has the biweight pq, and each partition is attached to a product of quantities b_{pq}, such that each factor corresponds to a single bipart in the partition.

Hence on proceeding to form a relation

$$c_{p_1q_1}^{\pi_1}\, c_{p_2q_2}^{\pi_2} \ldots = \ldots + \mathrm{P}\, b_{r_1s_1}^{\rho_1}\, b_{r_2s_2}^{\rho_2} \ldots + \ldots$$

wherein P represents the complete symmetric function cofactor of $b_{r_1s_1}^{\rho_1}\, b_{r_2s_2}^{\rho_2} \ldots$ it is clear that P is a linear function of symmetric function products, each of which has a specification

$$(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2} \ldots),$$

and is also a separation of the separable partition

$$(\overline{r_1s_1}^{\rho_1}\, \overline{r_2s_2}^{\rho_2} \ldots).$$

The partitions $(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2} \ldots)$, $(\overline{r_1s_1}^{\rho_1}\, \overline{r_2s_2}^{\rho_2} \ldots)$, as well as each of the separations which present themselves in the linear function P are of the same biweight.

When the separations in the function P are all expanded into a sum of monomial symmetric functions, each of the latter has the same biweight.

Taking the separable partition $(\overline{r_1s_1}^{\rho_1}\, \overline{r_2s_2}^{\rho_2} \ldots)$ as fixed, a definite number of specifications appertain to the separations. Forming then c products in correspondence with each of these specifications, the law of symmetry indicates that the *same number* of different monomial symmetric functions will appear in the developments of the several linear functions P. Further, the partitions of these monomial symmetric functions will be, in some order, identical with the several specifications of the separations of the fixed separable partition.

Assume the specifications to be, in any order

$$\theta_1,\, \theta_2,\, \ldots\, \theta_k,$$

and write the identity

$$c_{p_1q_1}^{\pi_1}\, c_{p_2q_2}^{\pi_2} \ldots = \ldots + \mathrm{P}\, b_{r_1s_1}^{\rho_1}\, b_{r_2s_2}^{\rho_2} \ldots + \ldots$$

in the abbreviated notation

$$c_\theta = \ldots + P_\theta b_r + \ldots$$

We have then the relations :—

$$c_{\theta_k} = \ldots + P_{\theta_k} b_r + \ldots \qquad (k = 1, 2, 3, \ldots k),$$

and

$$P_{\theta_1} = m_{11}\theta_1 + m_{12}\theta_2 + \ldots + m_{1k}\theta_k,$$
$$P_{\theta_2} = m_{21}\theta_1 + m_{22}\theta_2 + \ldots + m_{2k}\theta_k,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$P_{\theta_k} = m_{k1}\theta_1 + m_{k2}\theta_2 + \ldots + m_{kk}\theta_k,$$

the quantities m being numerical.

The determinant of the numbers m is symmetrical, for by the law of symmetry

$$m_{rs} = m_{sr}.$$

Hence, the coefficient of symmetric function θ_s in the development of the assemblage of separations P_{θ_r} is identical with the coefficient of symmetric function θ_r in the development of the assemblage of separations P_{θ_s}.

We may now proceed to express the symmetric functions θ as linear functions of the assemblages of separations P_θ and by elementary theory of determinants, the determinant of the system of results is symmetrical. Hence

$$\theta_1 = \mu_{11}P_{\theta_1} + \mu_{12}P_{\theta_2} + \ldots + \mu_{1k}P_{\theta_k},$$
$$\theta_2 = \mu_{21}P_{\theta_1} + \mu_{22}P_{\theta_2} + \ldots + \mu_{2k}P_{\theta_k},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\theta_k = \mu_{k1}P_{\theta_1} + \mu_{k2}P_{\theta_2} + \ldots + \mu_{kk}P_{\theta_k},$$

wherein

$$\mu_{rs} = \mu_{sr}.$$

36. From this are deduced two important theorems, the one a theorem of expressibility, and the other a theorem of symmetry. Any one of the monomial symmetric functions θ is expressed by a partition which is a specification of a separation of the partition $(\overline{r_1 s_1}^{\rho_1} \overline{r_2 s_2}^{\rho_2} \ldots)$. This implies that the biparts occurring in the partition of θ can be so partitioned into biparts that when assembled together they will be identical with the biparts of the partition $(\overline{r_1 s_1}^{\rho_1} \overline{r_2 s_2}^{\rho_2} \ldots)$. Hence the theorem of expressibility :—

37. *Theorem.*

"The biparts of the partition of a monomial symmetric function θ are partitioned in any manner into biparts, which, when all assembled together in a single bracket, are represented by

$$(\overline{r_1 s_1}^{\rho_1} \overline{r_2 s_2}^{\rho_2} \ldots).$$

The symmetric function θ is expressible as a linear function of assemblages of separations of the symmetric function $(\overline{r_1 s_1}^{\rho_1} \overline{r_2 s_2}^{\rho_2} \ldots)$."

38. The theorem of symmetry is as follows :—

Theorem.

"When the monomial symmetric function θ_r is expressed as a linear function of the assemblages of separations $P_{\theta_1}, P_{\theta_2}, \ldots P_{\theta_k}$, the coefficient of the assemblage P_{θ_s} is the same as the coefficient of the assemblage P_{θ_r} when θ_s is so expressed."

39. This theorem enables us to form a pair of symmetrical tables in regard to every partition of every biweight. The number of tables is therefore twice the number of partitions, the generating function for which has been already given.

§ 6. *The Functions composed of One Part.*

40. I will now establish a law by means of which any symmetric function expressed by a partition with a single bipart may be at once expressed in terms of separations of any partition of its biweight. It is merely necessary to interpret a result already obtained.

I recall the formula of Art. 26,

$$(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1,$$

which may also be written by Art. 8,

$$\Sigma_\pi \frac{(-)^{\Sigma\pi - 1}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} c_{p_1q_1}^{\pi_1} c_{p_2q_2}^{\pi_2} \ldots = (\overline{pq})\, \Sigma_\pi \frac{(-)^{\Sigma\pi - 1}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2} \ldots$$

Let us compare the cofactor of $b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2} \ldots$ in the development of the left hand side with its cofactor on the right hand side.

When the left hand side is multiplied out each symmetric function product which multiplies the term $b_{p_1q_1}^{\pi_1} b_{p_2q_2}^{\pi_2} \ldots$ is necessarily a separation of the symmetric function $(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)$. The result of the comparison will therefore be the expression of the function $(\overline{pq})$ in terms of such separations.

41. Let S_{pq} be the value assumed by c_{pq} when b_{pq} and other quantities b are put equal to unity.

Further, let $s_{(\overline{p_1q_1}^{\pi_1} \overline{p_2q_2}^{\pi_2} \ldots)}$ denote the expression of $s_{pq} = (\overline{pq})$ by means of separations of the symmetric function $(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)$.

Then we may write

$$\Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2! \ldots} s_{(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2! \ldots} S_{p_1 q_1}^{\pi_1} S_{p_2 q_2}^{\pi_2} \ldots$$

where S_{pq} denotes the sum of all the symmetric functions of biweight pq.

Represent the different separates of the partition $(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)$ by $(J_1), (J_2), \ldots$ and any separation by $(J_1)^{j_1} (J_2)^{j_2} \ldots$; substitute for the quantities S_{pq} their values in terms of symmetric functions; apply the multinominal theorem and equate corresponding portions of the two sides and there results the formula

$$(-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2! \ldots} s_{(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)} = \Sigma_j (-)^{\Sigma j - 1} \frac{(\Sigma j - 1)!}{j_1!\, j_2! \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots$$

where the summation is taken for every separation of the given partition.

42. This important result is a generalisation of the VANDERMONDE-WARING law for the expression of the sums of the powers of the roots of an equation in terms of the coefficients.

43. The formula may be reversed so as to exhibit any symmetric function whatever in terms of single bipart functions. The result easily reached is

$$(-)^{\Sigma\pi - 1} (\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)$$

$$= \Sigma_j (-)^{\Sigma j - 1} \frac{(\Sigma\pi_1 - 1)!\, (\Sigma\pi_2 - 1)! \ldots}{j_1!\, j_2! \ldots \pi_{11}!\, \pi_{12}! \ldots \pi_{21}!\, \pi_{22}! \ldots} s^{j_1}_{(\overline{p_{11} q_{11}}^{\pi_{11}} \overline{p_{12} q_{12}}^{\pi_{12}} \ldots)} s^{j_2}_{(\overline{p_{21} q_{21}}^{\pi_{21}} \overline{p_{22} q_{22}}^{\pi_{22}} \ldots)} \ldots,$$

the summation being for every separation

$$(\overline{p_{11} q_{11}}^{\pi_{11}} \overline{p_{12} q_{12}}^{\pi_{12}} \ldots)^{j_1} (\overline{p_{21} q_{21}}^{\pi_{21}} \overline{p_{22} q_{22}}^{\pi_{22}} \ldots)^{j_2} \ldots$$

of the symmetric function

$$(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots).$$

§ 7. *Second Law of Symmetry.*

44. The operation

$$g_{pq} = \partial_{a_{pq}} + a_{10} \partial_{a_{p+1,q}} + a_{01} \partial_{a_{p,q+1}} + \ldots + a_{rs} \partial_{a_{p+r,q+s}} + \ldots$$

may be said to be of biweight pq, since it lowers the weight of a symmetric function by the biweight pq. Further, its degree is zero, since it does not in general lower the degree of a symmetric function. If, however, g_{pq} operates upon a symmetric function of its own biweight, it is equivalent to the simple differential operation $\partial_{a_{pq}}$, and is of degree unity.

Similarly, the operation

$$\frac{\overline{g_{p_1q_1}^{\pi_1}\, g_{p_2q_2}^{\pi_2}\ldots}}{\pi_1!\,\pi_2!\ldots}$$

will be regarded as being of weight (biweight) pq, where $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ is a partition of the biweight pq; and if, as a particular case, the operand be of the same biweight pq, it will be equivalent to the operation

$$\frac{\partial_{a_{p_1q_1}}^{\pi_1}\,\partial_{a_{p_2q_2}}^{\pi_2}\ldots}{\pi_1!\,\pi_2!\ldots},$$

and will be of degree equal to

$$\pi_1+\pi_2+\ldots=\Sigma\pi.$$

Since therefore

$$\frac{\partial_{a_{p_1q_1}}^{\pi_1}\,\partial_{a_{p_2q_2}}^{\pi_2}\ldots}{\pi_1!\,\pi_2!}\,a_{p_1q_1}^{\pi_1}\,a_{p_2q_2}^{\pi_2}\ldots=1,$$

we have the result

$$\frac{\overline{g_{p_1q_1}^{\pi_1}\,g_{p_2q_2}^{\pi_2}\ldots}}{\pi_1!\,\pi_2!}\,a_{p_1q_1}^{\pi_1}\,a_{p_2q_2}^{\pi_2}\ldots=1;$$

assuming then a result

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_2=\ldots+\mathrm{P}b_{r_1s_1}^{\rho_1}\,b_{r_2s_2}^{\rho_2}\ldots+\ldots$$

derived from the three initial identities of Art. 24 and the relation assumed to exist between the quantities involved, we are at once led to the operator relation

$$\frac{\overline{g'^{\pi_1}_{p_1q_1}\,g'^{\pi_2}_{p_2q_2}\ldots}}{\pi_1!\,\pi_2!\ldots}=\ldots+\mathrm{P}\mathrm{G}''^{\rho_1}_{r_1s_1}\,\mathrm{G}''^{\rho_2}_{r_2s_2}\ldots+\ldots$$

where P consists entirely of symmetric functions of quantities which occur in the first identity. Further suppose a second result

$$(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)_2=\ldots+\mathrm{Q}b_{p_1q_1}^{\pi_1}\,b_{p_2q_2}^{\pi_2}\ldots+\ldots$$

Hence, operating on the left and right of this result with right and left sides of the foregoing operator relation, we obtain

$$(\ldots+\mathrm{P}\mathrm{G}''^{\rho_1}_{r_1s_1}\,\mathrm{G}''^{\rho_2}_{r_2s_2}\ldots+\ldots)\,(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$$
$$=\frac{\overline{g'^{\pi_1}_{p_1q_1}\,g'^{\pi_2}_{p_2q_2}\ldots}}{\pi_1!\,\pi_2!\ldots}(\ldots+\mathrm{Q}b_{p_1q_1}^{\pi_1}\,b_{p_2q_2}^{\pi_2}\ldots+\ldots),$$

or from theorems established above (Arts. 13, 44)

$$\mathrm{P}=\mathrm{Q},$$

no other terms surviving the operation.

45. Hence a theorem of symmetry :—

Theorem.

"If

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_2 = \ldots + \mathrm{P}b_{r_1s_1}^{\rho_1}\,b_{r_2s_2}^{\rho_2}\ldots + \ldots,$$

the cofactor symmetric function P is unaltered when the partitions $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$, are interchanged."

The function P presents itself in the first place as a linear function of separations of the partition of the b product to which it is attached. The theorem supplies linear functions of separations of any two partitions $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$ respectively, of the same biweight, which are equal to one another.

46. To make the matter clear, form a table of biweight 21 as follows :—

	b_{21}	$b_{20}b_{01}$	$b_{11}b_{10}$	$b_{10}^2b_{01}$
$(\overline{21})_2$	$(\overline{21})$	$(\overline{20}\,\overline{01})$ $-(\overline{20})(\overline{01})$	 $-(\overline{11})(\overline{10})$	$(\overline{10}^2\,\overline{01})$ $-(\overline{10}^2)(\overline{01})$ $-(\overline{10}\,\overline{01})(\overline{10})$ $+(\overline{10})^2(\overline{01})$
$(\overline{20}\,\overline{01})_2$	$-(\overline{21})$	$-(\overline{20}\,\overline{01})$ $-(\overline{20})(\overline{01})$	$-(\overline{11}\,\overline{10})$ $+(\overline{11})(\overline{10})$	$-(\overline{10}^2\,\overline{01})$ $-(\overline{10}^2)(\overline{01})$ $+(\overline{10}\,\overline{01})(\overline{10})$
$(\overline{11}\,\overline{10})_2$	$-(\overline{21})$	$-(\overline{20}\,\overline{01})$ $+(\overline{20})(\overline{01})$	$-(\overline{11}\,\overline{10})$	$-(\overline{10}^2\,\overline{01})$ $+(\overline{10}^2)(\overline{01})$
$(\overline{10}^2\,\overline{01})_2$	$(\overline{21})$	$(\overline{20}\,\overline{01})$	$(\overline{11}\,\overline{10})$	$(\overline{10}^2\,\overline{01})$

which is to be read by rows.*

Each term in a column is a separation of the partition of the b product at the head of the column.

The separations in each line of terms as written possess the same specifications, and also the same numerical coefficients. In the right hand column the partition separated is a fundamental symmetric function, and hence each separate therein appearing is so also. Each block of separations in the right hand column is the expression by means of fundamental symmetric functions of the monomial symmetric function of the same elements whose partition appears to the left of the same line. The terms of the first three columns may be regarded as being formed according to the same *law* as the right hand column, and therefore according to a law defined by

* Each term in the left hand column is equal to the aggregate of terms in *any* block in the same row.

the monomial function at the left of the same line. For example, the terms in the second column and third line are separations of $(\overline{20}\,\overline{01})$ formed according to the law of the function $(\overline{11}\,\overline{10})$. Also the terms in the third column and second line are separations of $(\overline{11}\,\overline{10})$ formed according to the law of the function $(\overline{20}\,\overline{01})$. Now observe that the law of symmetry establishes that the table enjoys row and column symmetry. Hence the assemblage of separations of $(\overline{20}\,\overline{01})$ formed according to the law of $(\overline{11}\,\overline{10})$ is equal to the assemblage of separations of $(\overline{11}\,\overline{10})$ formed according to the law of $(\overline{20}\,\overline{01})$.

47. Hence in general the theorem :—

"The assemblage of separations of $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$, formed according to the law of $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, is equal to the assemblage of separations of $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, formed according to the law of $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$."

In the particular case considered the equality is

$$-(\overline{20}\,\overline{01})+(\overline{20})(\overline{01})=-(\overline{11}\,\overline{10})+(\overline{11})(\overline{10}).$$

To actually form separations of $(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)$, according to the law of $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, the separations of the former must be written down, and also the expression of the latter, by means of fundamental symmetric functions. The separations are then given the same coefficients as the products of fundamental symmetric functions which possess the same specifications.

§ 8. *Third Law of Symmetry.*

48. From the relation

$$c_{p_1q_1}^{\pi_1}\,c_{p_2q_2}^{\pi_2}\ldots=\ldots+\mathrm{L}b_{r_1s_1}^{\rho_1}\,b_{r_2s_2}^{\rho_2}\ldots+\ldots$$

is derived the operator relation

$$\mathrm{G}'^{\pi_1}_{p_1q_1}\,\mathrm{G}'^{\pi_2}_{p_2q_2}\ldots=\ldots+\mathrm{L}\mathrm{G}''^{\rho_1}_{r_1s_1}\,\mathrm{G}''^{\rho_2}_{r_2s_2}\ldots+\ldots$$

and, thence, by the method already employed,

$$(\overline{r_1s_1}^{\rho_1}\,\overline{r_2s_2}^{\rho_2}\ldots)_2=\ldots+\mathrm{L}(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)_1+\ldots$$

This law of symmetry is of considerable importance and interest, but I do not stop to further discuss it.*

* *Vide* 'American Journal of Mathematics.' "Third Memoir on a New Theory of Symmetric Functions," now in progress in vols. 11, 12, and succeeding volumes.

§ 9. *The linear Partial Differential Operations of the Theory of Separations.*

49. For purposes of calculation it is necessary to adapt the operations

$$g_{10},\ g_{01},\ \ldots\ g_{pq}\ \ldots,$$

so that they may be performed on a symmetric function when the latter is expressed in terms of the separations of any given partition.

Of any partition $(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ldots)$ separates (*vide* Definitions) are formed by taking all possible combinations of the parts. These are precisely

$$(\pi_1+1)(\pi_2+1)\ldots-1,$$

distinct separates which must be regarded as independent variables.

Put

$$(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)$$

for any separate of a given separable partition P.

Then by a known theorem

$$g_{pq}=\Sigma g_{pq}(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)},$$

the summation being in regard to all the separates.

Moreover (Art. 17)

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}=\Sigma\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!}\mathrm{G}_{10}^{\pi_{10}}\mathrm{G}_{01}^{\pi_{01}}\ldots\mathrm{G}_{p_1q_1}^{\pi_{p_1q_1}}\ldots,$$

the summation being in regard to all the partitions $(\overline{10}^{\pi_{10}}\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)$ of the biweight pq; and also (Art. 13)

$$\mathrm{G}_{10}^{\pi_{10}}\mathrm{G}_{01}^{\pi_{01}}\ldots\mathrm{G}_{p_1q_1}^{\pi_{p_1q_1}}\ldots(\overline{10}^{\pi_{10}+\rho_{10}}\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)=(\overline{10}^{\rho_{10}}\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots).$$

50. Hence

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!\ldots}g_{pq}$$
$$=\Sigma_\pi\Sigma_\rho\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!\ldots}(\overline{10}^{\rho_{10}}\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{01}}\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)}$$

the summation being in regard to

(1) Every separate of the given separable partition;

(2) Every partition of the biweight pq.

The right hand side of this relation may be broken up into fragments in each of which all the numbers $\pi_{10}, \pi_{01}, \ldots \pi_{p_1q_1} \ldots$ are constant.

In fact we may write

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}$$

$$= \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!\ldots}\Sigma_\rho\,(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)},$$

wherein, following the summation sign Σ_ρ, the numbers $\pi_{10}, \pi_{01}, \ldots \pi_{p_1q_1} \ldots$ are constant, and the operator

$$\Sigma_\rho\,(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)},$$

is one of the fragments above mentioned.

This operation has a biweight pq, and may be defined also in regard to the partition $(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)$ of the biweight pq.

51. Write then for brevity and convenience

$$\Sigma_\rho\,(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)\,\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)} = g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)};$$

so that we may write

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!\ldots}g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)},$$

where the summation is in regard to every partition $(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)$ of the biweight pq.

52. In general not every partition of the biweight pq will occur in the given separable partition, but it is convenient to consider the general result just written down as including every such partition. It will be seen later that this result is of great importance in the theory.

I remark that on the left-hand side we have a linear partial differential operation g_{pq} whose expression by means of the fundamental symmetric functions and their differential inverses is well known by what has preceded. Such expression is all that is needed so long as we are concerned only with the fundamental forms which, as they appear in the expression of a monomial symmetric function of biweight pq, present themselves in products which are separations of the symmetric function $(\overline{10}^p\,\overline{01}^q)$. In the present broader theory in which the leading idea is the consideration of any partition at pleasure of the biweight as the separable partition, we bring into view the exhibition of the operation g_{pq} as a linear function of operations, each of which is in correspondence with a partition of the biweight.

We have, in fact, a biweight operator g_{pq} decomposed into a full number of bipart partition operators

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}.$$

Observe that the whole theory of separations is a generalisation from a weight to a partition of a weight. Here we have generalised from a weight operation to a partition operation, and I henceforward regard the partition operator as the essential linear partial differential operator of the theory. The biweight operator g_{pq} has been expressed *ante* (Art. 17) in terms of the obliterating operators of the form G_{pq}. These operations G_{pq} are equally available in the theory of the separable partition in general. The mode of their operation upon a symmetric function product will be subsequently explained (in § 11). So far I have merely considered their operation upon monomial forms.*

53. I observe that the biweight operator g_{pq} is expressed as a linear function of the partition operators of the same biweight, according to the same laws as—

(1) The operator g_{pq} is expressed in terms of the operations G_{pq} (Art. 17).

(2) The symmetric functions, containing one part only, are expressed in terms of the fundamental or single-unitary forms (Art. 8).

E.g., compare the three results (the first slightly modified) :—

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}=\Sigma_\pi\,\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)},$$

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq}=\Sigma_\pi\,\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}G_{p_1q_1}^{\pi_1}\,G_{p_2q_2}^{\pi_2}\ldots,$$

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}(\overline{pq})=\Sigma_\pi\,\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}a_{p_1q_1}^{\pi_1}\,a_{p_2q_2}^{\pi_2}\ldots.$$

54. For convenience of reference, I write down the particular simplest cases of the decomposition.

$$g_{10}=g_{(\overline{10})},$$
$$g_{01}=g_{(\overline{01})},$$
$$g_{20}=g_{(\overline{10}^2)}-2g_{(\overline{20})},$$
$$g_{11}=g_{(\overline{10}\,\overline{01})}-g_{(\overline{11})},$$
$$g_{02}=g_{(\overline{01}^2)}-2g_{(\overline{02})},$$
$$g_{30}=g_{(\overline{10}^3)}-3g_{(\overline{20}\,\overline{10})}+3g_{(\overline{30})},$$
$$g_{21}=g_{(\overline{10}^2\,\overline{01})}-g_{(\overline{20}\,\overline{10})}-g_{(\overline{11}\,\overline{10})}+g_{(\overline{21})},$$
$$g_{12}=g_{(\overline{01}^2\,\overline{10})}-g_{(\overline{02}\,\overline{10})}-g_{(\overline{11}\,\overline{01})}+g_{(\overline{12})},$$
$$g_{03}=g_{(\overline{01}^3)}-3g_{(\overline{02}\,\overline{01})}+3g_{(\overline{03})}.$$

* The decomposition of the obliterating operation G_{pq} into partition obliterating operations is given *post* § 10.

55. I also give the developed expressions of a few of the partition operators. Thus

$$g_{(\overline{10})} = \partial_{(\overline{10})} + (\overline{10})\,\partial_{(\overline{10}^2)} + (\overline{01})\,\partial_{(\overline{10}\,\overline{01})} + (\overline{20})\,\partial_{(\overline{20}\,\overline{10})} + (\overline{10}^2)\,\partial_{(\overline{10}^3)}$$
$$+ (\overline{11})\,\partial_{(\overline{11}\,\overline{10})} + (\overline{10}\,\overline{01})\,\partial_{(\overline{10}^2\,\overline{01})} + (\overline{02})\,\partial_{(\overline{10}\,\overline{02})} + (\overline{01}^2)\,\partial_{(\overline{10}\,\overline{01}^2)}$$
$$+ \ldots,$$

$$g_{(\overline{20}\,\overline{01})} = \partial_{(\overline{20}\,\overline{01})} + (\overline{10})\,\partial_{(\overline{20}\,\overline{10}\,\overline{01})} + (\overline{01})\,\partial_{(\overline{20}\,\overline{01}^2)} + (\overline{20})\,\partial_{(\overline{20}^2\,\overline{01})}$$
$$+ (\overline{10}^2)\,\partial_{(\overline{20}\,\overline{10}^2\,\overline{01})} + (\overline{11})\,\partial_{(\overline{20}\,\overline{11}\,\overline{01})} + (\overline{10}\,\overline{01})\,\partial_{(\overline{20}\,\overline{10}\,\overline{01}^2)}$$
$$+ (\overline{02})\,\partial_{(\overline{20}\,\overline{01}\,\overline{02})} + (\overline{01}^2)\,\partial_{(\overline{20}\,\overline{01}^3)} + \ldots$$

The mode of operation of the biweight operators in the separation theory is now manifest.

56. Let

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}, \quad g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\,\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)},$$

be any two partition operators of the same or different biweights. Representing them for brevity by

$$g_{(\pi)}, \quad g_{(\rho)},$$

we have

$$g_{(\pi)}g_{(\rho)} = \overline{g_{(\pi)}g_{(\rho)}} + g_{(\pi)} \dagger g_{(\rho)},$$

wherein the multiplication on the left denotes successive operation, the bar on the right denotes symbolical multiplication, and the symbol $\dagger$ denotes explicit differentiation on the operand regarded as a function of symbols of quantity only.

It is easy to establish the result

$$g_{(\pi)} \dagger g_{(\rho)} = g_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)}$$
$$= g_{(\pi+\rho)} \text{ for brevity.}$$

Hence

$$g_{(\pi)} \dagger g_{(\rho)} = g_{(\rho)} \dagger g_{(\pi)} = g_{(\pi+\rho)},$$

and

$$g_{(\pi)}\,g_{(\rho)} = \overline{g_{(\pi)}\,g_{(\rho)}} + g_{(\pi+\rho)},$$

or at full length

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}\,g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\,\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)} = \overline{g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}\,g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\,\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)}},$$
$$+ g_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}+\rho_{p_1q_1}}\ldots)},$$

the fundamental law of multiplication (compare Art. 15).

Also

$$g_{(\pi)}\,g_{(\rho)} - g_{(\rho)}\,g_{(\pi)} = 0,$$

or

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)}\,g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\,\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)} - g_{(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\,\overline{p_1q_1}^{\rho_{p_1q_1}}\ldots)}\,g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)} = 0,$$

shewing that any two partition operators are commutative.

57. The left hand side of the result just reached is called by SOPHUS LIE* the "Zusammensetzung" or "Combination" of the operators which appear. SYLVESTER† has also called it the 'Alternant' of the two operators.

The whole system of partition operators forms an infinite group in co-relation with an infinite group of transformations—

We can state the theorem :—

Theorem.

"The Combination or Alternant of any two partition operators vanishes."

Considering the partial differential equation

$$g_{(\pi)} = 0,$$

and ϕ any function which is a solution, then must $g_{(\rho)}\phi$ be also a solution, since

$$g_{(\pi)}\,g_{(\rho)}\,\phi - g_{(\rho)}\,g_{(\pi)}\,\phi = 0.$$

Theorem.

"If $g_{(\pi)}$ and $g_{(\rho)}$ be any two partition operators, and ϕ a solution of the equation $g_{(\pi)} = 0$; then will $g_{(\rho)}\,\phi$ be also a solution of the same equation."

58. Consider now the partial differential equation of Art. 51,

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq} = \Sigma_\pi\,\frac{(-)^{\Sigma\pi-1}\,(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1q_1}!\ldots}\,g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)} = 0.$$

Assume the separable partition to be

$$(\overline{10}^{\sigma_{10}}\,\overline{01}^{\sigma_{01}}\ldots\overline{p_1q_1}^{\sigma_{p_1q_1}}\ldots),$$

so that the operand is a linear function of separations of this partition.

The effect of the partition operator

$$g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\,\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)},$$

is the production of terms each of which is a separation of the partition

$$(\overline{10}^{\sigma_{10}-\pi_{10}}\,\overline{01}^{\sigma_{01}-\pi_{01}}\ldots\overline{p_1q_1}^{\sigma_{p_1q_1}-\pi_{p_1q_1}}\ldots).$$

* 'Theorie der Transformationsgruppen,' Leipzig, 1888.

† 'Lectures on the Theory of Reciprocants.' ('American Journal of Mathematics,' and elsewhere.)

Observe that separations of this partition cannot be produced by any *other* of the partition operators which present themselves on the left hand side of the differential equation. Hence if the operand satisfies the differential equation

$$g_{pq} = 0,$$

it must also satisfy the differential equation

$$g_{(\overline{10}^{\pi_{10}}\;\overline{01}^{\pi_{01}}\;\ldots\;\overline{p_1q_1}^{\pi_{p_1q_1}}\ldots)} = 0.$$

59. This important theorem may be enunciated as follows:—

Theorem.

"If a function, expressed in terms of separations of a given monomial symmetric function, be annihilated by a biweight operator it must also be annihilated by every partition operator of that biweight."

As regards the calculation of Tables of Separations of Symmetric Functions, this is the cardinal theorem.

As an example of its application I propose to utilise it for the purpose of exhibiting the function $(\overline{31}\;\overline{01})$ as a linear function of separations of $(\overline{21}\;\overline{10}\;\overline{01})$. The law of expressibility shows this to be possible, for $(\overline{21}\;\overline{10})$ is a partition of the biweight 31. Remarking that the separation $(\overline{21})\,(\overline{10})\,(\overline{01})$ cannot occur in the expression, since it is the only separation which produces the monomial $(\overline{32})$ when multiplied out, I put

$$(\overline{31}\;\overline{01}) = \mathrm{A}\,(\overline{21}\;\overline{10})\,(\overline{01}) + \mathrm{B}\,(\overline{21}\;\overline{01})\,(\overline{10}) + \mathrm{C}\,(\overline{10}\;\overline{01})\,(\overline{21}) + \mathrm{D}\,(\overline{21}\;\overline{10}\;\overline{01}).$$

A monomial symmetric function is caused to vanish by means of the operation of the biweight operator g_{pq} if no partition of the biweight pq is comprised amongst its parts. In consequence of this, the only biweight operators which do not cause it to vanish are g_{31}, g_{01}, and g_{32}. Hence all the partition operators of every other biweight operator annihilate the function $(\overline{31}\;\overline{01})$. It suffices to employ as annihilators the two partition operators $g_{(\overline{01})}$ and $g_{(\overline{21})}$.

Hence, retaining only significant terms,

$$\{\partial_{(\overline{10})} + (\overline{01})\,\partial_{(\overline{10}\;\overline{01})} + (\overline{21})\,\partial_{(\overline{21}\;\overline{10})} + (\overline{21}\;\overline{01})\,\partial_{(\overline{21}\;\overline{10}\;\overline{01})}\}\,(\overline{31}\;\overline{01}) = 0,$$

$$\{\partial_{(\overline{21})} + (\overline{10})\,\partial_{(\overline{21}\;\overline{10})} + (\overline{01})\,\partial_{(\overline{21}\;\overline{01})} + (\overline{10}\;\overline{01})\,\partial_{(\overline{21}\;\overline{10}\;\overline{01})}\}\,(\overline{31}\;\overline{01}) = 0,$$

leading to

$$\mathrm{A} + \mathrm{C} = 0,\quad \mathrm{B} + \mathrm{D} = 0,$$
$$\mathrm{C} + \mathrm{D} = 0,\quad \mathrm{A} + \mathrm{B} = 0,$$

or

$$\mathrm{D} = -\,\mathrm{C} = -\,\mathrm{B} = \mathrm{A}.$$

Hence

$$(\overline{31}\,\overline{01}) = \mathrm{A}\,\{(\overline{21}\,\overline{10})\,(\overline{01}) - (\overline{21}\,\overline{01})\,(\overline{10}) - (\overline{10}\,\overline{01})\,(\overline{21}) + (\overline{21}\,\overline{10}\,\overline{01})\}\,;$$

and it is easy to see that $\mathrm{A} = -\frac{1}{2}$, for each of the products $(\overline{21}\,\overline{01})\,(\overline{10})$, $(\overline{10}\,\overline{01})\,(\overline{21})$ on multiplication produces a term $+\,(\overline{31}\,\overline{01})$, and this monomial is not produced by the development of either of the other two products. The value of A may, however, be instructively obtained by means of the operator g_{01}.

For

$$g_{01}\,(\overline{31}\,\overline{01}) = (\overline{31}),$$

and

$$g_{01} = g_{(\overline{01})} = \partial_{(\overline{01})} + (\overline{10})\,\partial_{(\overline{10}\,\overline{01})} + (\overline{21})\,\partial_{(\overline{21}\,\overline{01})} + (\overline{21}\,\overline{10})\,\partial_{(\overline{21}\,\overline{10}\,\overline{01})}. \quad \text{(Art. 53.)}$$

Hence

$$(\overline{31}) = 2\mathrm{A}\,\{-\,(\overline{21})\,(\overline{10}) + (\overline{21}\,\overline{10})\},$$

and now A is obviously equal to $-\frac{1}{2}$.

But we may further employ the operator g_{31}.

For

$$g_{31} = -\,\mathrm{G}_{31} + \ldots = +\,g_{(\overline{21}\,\overline{10})} + \ldots \quad \text{(Arts. 17, 53)},$$

significant terms only being retained; hence $-\,\mathrm{G}_{31}$ and $g_{(\overline{21}\,\overline{10})} \equiv \partial_{(\overline{21}\,\overline{10})}$ are equivalent operations in the present case, and performing them on their own sides $-1 = 2\mathrm{A}$ or $\mathrm{A} = -\frac{1}{2}$.

Thus

$$(\overline{31}\,\overline{01}) = -\tfrac{1}{2}\,(\overline{21}\,\overline{10})\,(\overline{01}) + \tfrac{1}{2}\,(\overline{21}\,\overline{01})\,(\overline{10}) + \tfrac{1}{2}\,(\overline{10}\,\overline{01})\,(\overline{21}) - \tfrac{1}{2}\,(\overline{21}\,\overline{10}\,\overline{01}).$$

§ 10. *The Partition obliterating Operators.*

60. In the foregoing a generalisation has been made from a number to the partition of a number in the case of the operations $g_{10}, g_{01}, \ldots g_{pq}, \ldots$ The possibility of the like generalisation in respect of the obliterating operators $\mathrm{G}_{10}, \mathrm{G}_{01}, \ldots \mathrm{G}_{pq}, \ldots$ is naturally presented as a subject for enquiry.

Consider a symmetric function

$$f\,(\alpha_{10}, \alpha_{01}, \ldots \alpha_{pq}, \ldots) = f$$

to be the product of m monomial functions, and write

$$f = f_1 f_2 \ldots f_m.$$

Supposing a_{pq} changed into $a_{pq} + \mu a_{p-1,\,q} + \nu a_{p,\,q-1}$, we have from previous work

$$\begin{aligned}&(1 + \mu G_{10} + \nu G_{01} + \ldots\ldots + \mu^p\nu^q G_{pq} + \ldots) f\\ &= (1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots) f_1\\ &\quad\times (1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots) f_2\\ &\quad\times \ldots \qquad\qquad \vdots\\ &\quad\times (1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots) f_m.\end{aligned}$$

Expanding the right hand side and equating coefficients of like products of powers μ and ν, we get

$$\begin{aligned}G_{10}f &= \Sigma(G_{10}f_1) f_2 f_3 \ldots f_m,\\ G_{20}f &= \Sigma(G_{10}f_1)(G_{10}f_2) f_3 \ldots f_m + \Sigma(G_{20}f_1) f_2 f_3 \ldots f_m,\\ G_{11}f &= \Sigma(G_{10}f_1)(G_{01}f_2) f_3 \ldots f_m + \Sigma(G_{11}f_1) f_2 f_3 \ldots f_m,\\ G_{30}f &= \Sigma(G_{10}f_1)(G_{10}f_2)(G_{10}f_3) f_4 \ldots f_m + \Sigma(G_{20}f_1)(G_{10}f_2) f_3 \ldots f_m + \Sigma(G_{30}f_1) f_2 \ldots f_m,\\ G_{21}f &= \Sigma(G_{10}f_1)(G_{10}f_2)(G_{01}f_3) f_4 \ldots f_m + \Sigma(G_{11}f_1)(G_{10}f_2) f_3 \ldots f_m,\\ &= \Sigma(G_{20}f_1)(G_{01}f_2) f_3 \ldots f_m + \Sigma(G_{21}f_1) f_2 f_3 \ldots f_m,\end{aligned}$$

and so forth, where the summations are, in regard to the different terms, obtained by permutation of the m suffixes of the functions $f_1, f_2, \ldots f_m$.

In general in the expression for $G_{pq}f$ there will occur a summation corresponding to each partition of the biweight pq. If a partition be $(\overline{p_1q_1}\,\overline{p_2q_2} \ldots \overline{p_sq_s})$ the summation is

$$\Sigma(G_{p_1q_1}f_1)(G_{p_2q_2}f_2) \ldots (G_{p_sq_s}f_s) f_{s+1} \ldots f_m.$$

61. Thus, when performed upon a product of functions, the operator G_{pq} breaks up into as many distinct operations as the biweight pq possesses partitions. It is convenient to denote the operation indicated by the summation

$$\Sigma(G_{p_1q_1}f_1)(G_{p_2q_2}f_2) \ldots (G_{p_sq_s}) f_s f_{s+1} \ldots f_m,$$

by

$$G_{(\overline{p_1q_1}\,\overline{p_2q_2} \ldots \overline{p_sq_s})},$$

and to speak of it as a partition operator.

62. We may now write down an equivalence

$$G_{pq} = \Sigma G_{(\overline{p_1q_1}\,\overline{p_2q_2} \ldots \overline{p_sq_s})},$$

where the summation is in regard to every partition of the biweight pq. This is, in fact, a theorem for operating with G_{pq} upon a product of symmetric functions, and it is consistent with the more simple law previously established.

In particular

$$G_{10} = G_{(\overline{10})},$$
$$G_{01} = G_{(\overline{01})},$$
$$G_{20} = G_{(\overline{10}^2)} + G_{(\overline{20})},$$
$$G_{11} = G_{(\overline{10}\,\overline{01})} + G_{(\overline{11})},$$
$$G_{02} = G_{(\overline{01}^2)} + G_{(\overline{02})}.$$
$$\vdots$$

63. The relations between the partition g operators and the partition G operators are of great interest.

Recalling the equivalence (Arts. 53 and 41)

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots} g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\dots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots} G_{p_1q_1}^{\pi_1} G_{p_2q_2}^{\pi_2}\dots,$$

which should be compared with the algebraical result

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots} s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\dots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots} S_{p_1q_1}^{\pi_1} S_{p_2q_2}^{\pi_2}\dots,$$

there arise the relations :—

$$g_{(\overline{10})} = G_{10} = G_{(\overline{10})},$$
$$g_{(\overline{01})} = G_{01} = G_{(\overline{01})},$$
$$g_{(\overline{10}^2)} - 2g_{(\overline{20})} = G_{10}^2 - 2G_{20} = G_{(\overline{10})}^2 - 2G_{(\overline{10}^2)} - 2G_{(\overline{20})},$$
$$g_{(\overline{10}\,\overline{01})} - g_{(\overline{11})} = G_{10}G_{01} - G_{11} = G_{(\overline{10})}G_{(\overline{01})} - G_{(\overline{10}\,\overline{01})} - G_{(\overline{11})},$$
$$g_{(\overline{10}^3)} - 3g_{(\overline{20}\,\overline{10})} + 3g_{(\overline{30})} = G_{10}^3 - 3G_{20}G_{10} + 3G_{30},$$
$$= G_{(\overline{10})}^3 - 3G_{(\overline{10}^2)}G_{(\overline{10})} + 3G_{(\overline{10}^3)} - 3\{G_{(\overline{20})}G_{(\overline{10})} - G_{(\overline{20}\,\overline{10})}\} + 3G_{(\overline{30})},$$

and so forth.

64. Now consider the relation last written.

I say that it may be broken up into three relations, viz. :—

$$g_{(\overline{10}^3)} = G_{(\overline{10})}^3 - 3G_{(\overline{10}^2)}G_{(\overline{10})} + 3G_{(\overline{10}^3)},$$
$$g_{(\overline{20}\,\overline{10})} = G_{(\overline{20})}G_{(\overline{10})} - G_{(\overline{20}\,\overline{10})},$$
$$g_{(\overline{30})} = G_{(\overline{30})};$$

for suppose an operand to be composed of separations of a separable partition $(\overline{10}^{\pi_{10}}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}}\dots)$, the performance of the operations on the two sides of the relation produces the same result identically. This result is composed of three portions containing separations of the partitions

$$(\overline{10}^{\pi_{10}-3}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}}\ldots),\qquad(\overline{10}^{\pi_{10}-1}\,\overline{20}^{\pi_{20}-1}\,\overline{30}^{\pi_{30}}\ldots),\qquad(\overline{10}^{\pi_{10}}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}-1}\ldots)$$

respectively. Hence the operations which produce the three identically equal portions of the result must be equivalent, and the three relations between the operators therefore follow.

In general, we may say that of the biweight pq there are as many relations between partition operations as there are partitions of the biweight pq.

65. The general law of the coefficients will be now investigated.

In the result of Art. 53, viz.,

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} G_{p_1q_1}^{\pi_1} G_{p_1q_1}^{\pi_2}\ldots,$$

we have to substitute for $G_{p_1q_1}$, $G_{p_2q_2}\ldots$, the sums of the partition G operators of weights p_1q_1, $p_2q_2\ldots$, respectively; we have then to collect on the right all the G products which are associated with separations of one and the same partition, and to equate them to the corresponding g operator on the left. It is evident that this process does not alter the *law* of the coefficients, and that representing the different separates of the given partition by (J_1), $(J_2)\ldots$, and any separation by $(J_1)^{j_1}(J_2)^{j_2}\ldots$, we may write

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots} G_{(J_1)}^{j_1} G_{(J_2)}^{j_2}\ldots$$

Observe that this is precisely the law which gives the expression of a single bipart function in terms of separations of the partition $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$. I recall the result of Art. 41, viz.,

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots,$$

which renders the correspondence between the algebraic and differential theories very striking.

66. Reversing the formula as in the algebraic theory we get the important formula :—

$$(-)^{\Sigma\pi-1}G_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)}$$
$$= \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma\pi_1-1)!\,(\Sigma\pi_2-1)!\ldots}{j_1!\,j_2!\ldots\pi_{11}!\,\pi_{12}!\ldots\pi_{21}!\,\pi_{22}!\ldots} g_{(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}^{j_1}\, g_{(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}^{j_2}\ldots,$$

the summation being for every separation

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots$$

of the partition $(\overline{p_1q_1}^{\pi_1}\overline{p_2q_2}^{\pi_2}\ldots)$. (Compare Art. 43.)

Observe that in these formulæ the multiplications of operations are non-symbolic and denote successive operations.

67. Remark the results of operations :—

$$G_{(\overline{p_1q_1}^{\pi_1}\ \overline{p_2q_2}^{\pi_2}\ldots)}\, s_{p_1q_1}^{\ \pi_1}\, s_{p_2q_2}^{\ \pi_2}\ldots = 1,$$

$$\frac{1}{j_1!}\frac{1}{j_2!}\ldots \overset{j_1}{g}_{(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}\overset{j_2}{g}_{(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}\ldots$$
$$\ldots(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots = 1.$$

§ 11. *The Multiplication of Symmetric Functions.*

68. The partition G operators are of great service in multiplication. An example will make this clear. It is required to find the coefficient of $(\overline{11}^2)$ in the product $(\overline{10}^2)(\overline{01})^2$.

Put

$$(\overline{10}^2)(\overline{01})^2 = \ldots + A\,(\overline{11}^2) + \ldots$$

On operating with $\overset{2}{G}_{11}$ on the right the result is A, since every other term is annihilated ; and since

$$G_{11} = G_{(\overline{11})} + G_{(\overline{10}\ \overline{01})},$$

we have

$$\overset{2}{G}_{11}(\overline{10}^2)(\overline{01})^2 = \{G_{(\overline{11})} + G_{(\overline{10}\ \overline{01})}\}^2(\overline{10}^2)(\overline{01})^2 = A,$$

therefore

$$\{G_{(\overline{11})} + G_{(\overline{10}\ \overline{01})}\}\,.\,2\,(\overline{10})(\overline{01}) = A,$$

hence

$$A = 2.$$

Similarly putting

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)(p_{21}q_{21}^{\pi_{21}}\ p_{22}q_{22}^{\pi_{22}}\ldots)(\ldots) = \ldots + A\,(\overline{r_1s_1}^{\rho_1}\ \overline{r_2s_2}^{\rho_2}\ldots) + \ldots$$

we have merely to operate on the left-hand side with the partition operators equivalent to $\overset{\rho_1}{G}_{r_1s_1}\overset{\rho_2}{G}_{r_2s_2}\ldots$ in order to find A.

§ 12. *Symmetric Functions of Differences.*

69. In the unipartite theory there is a transformation which connects the Symmetric Functions of the Differences of the roots of the equation

$$x^n - na_1x^{n-1} + n(n-1)a_2x^{n-2} - \ldots + (-)^n n!\, a_n = 0$$

with the non-unitary symmetric functions of the roots of the equation

$$x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots + (-)^n a_n = 0. \qquad (n = \infty)$$

In fact, the annihilating operator is in each case found to be

$$g_1 = \partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \ldots$$

The theory of the Invariants and Covariants of a Binary quantic may be thus brought to depend upon non-unitary symmetric functions. (*Vide* 'American Journal of Mathematics,' vol. 6, p. 131.)

In the present case, there is also a transformation. For the purpose in hand, write the fundamental identity in the form

$$(1 + \alpha_1x + \beta_1y)(1 + \alpha_2x + \beta_2y) \ldots (1 + \alpha_nx + \beta_ny)$$
$$= 1 + na_{10}x + na_{01}y + \ldots + \frac{n!}{(n-p-q)!} a_{pq}x^py^q + \ldots$$

Any function of the differences of the quantities on the left remains unaltered, when we write for the quantities α_s, β_s, respectively $\alpha_s + h$ and $\beta_s + h$. The coefficient of x^py^q on the right then becomes

$$\Sigma(\alpha_1 + h)(\alpha_2 + h) \ldots (\alpha_p + h)(\beta_{p+1} + h)(\beta_{p+2} + h) \ldots (\beta_{p+q} + h),$$

which is

$$(\overline{10}^p\, \overline{01}^q) + (n-p-q+1)\{(\overline{10}^{p-1}\, \overline{01}^q) + (\overline{10}^p\, \overline{01}^{q-1})\}h$$
$$+ \frac{(n-p-q+1)(n-p-q+2)}{2!}\{(\overline{10}^{p-2}\, \overline{01}^q) + 2(\overline{10}^{p-1}\, \overline{01}^{q-1}) + (\overline{10}^p\, \overline{01}^{q-2})\}h^2$$
$$+ \ldots$$

But

$$(\overline{10}^p\, \overline{01}^q) = \frac{n!}{(n-p-q)!} a_{pq}.$$

Hence a_{pq} is transformed into

$$a_{pq} + (a_{p-1,q} + a_{p,q-1})h + (a_{p-2,q} + 2a_{p-1,q-1} + a_{p,q-2})\frac{h^2}{2!} + \ldots,$$

the general term being

$$\frac{a_{p-s,\, q-t}\, h^{s+t}}{s!\, t!}.$$

Hence any symmetrical function

$$f(a_{10}, a_{01}, \ldots a_{pq} \ldots) \equiv f,$$

is transformed into

$$f\{a_{10} + h,\ a_{01} + h, \ldots a_{pq} + (a_{p-1,\,q} + a_{p,\,q-1})\,h + \ldots\ldots\},$$

or writing

$$g_{pq} = \partial_{a_{pq}} + a_{10}\partial_{a_{p+1,\,q}} + a_{01}\partial_{a_{p,\,q+1}} + \ldots,$$

this is

$$\overline{\exp}\left\{(g_{10} + g_{01})\,h + (g_{20} + 2g_{11} + g_{02})\frac{h^2}{2!} + \ldots\right\}f,$$

the bar over exp denoting that the multiplications of operators, which arise, are symbolic.

Now, by the theorem of Art. 15, this is

$$\exp\{M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots\}f,$$

the multiplication denoting successive operations, and identically

$$\begin{aligned}\exp(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots)\\ = 1 + \xi + \eta + \frac{1}{2!}(\xi + \eta)^2 + \frac{1}{3!}(\xi + \eta)^3 + \ldots\\ = \exp(\xi + \eta).\end{aligned}$$

Hence $M_{10} = M_{01} = 1$ and the other coefficients M are zero.

Hence the symmetric function f is converted into

$$\exp(g_{10} + g_{01})\,h\,.\,f,$$

and, if f be a function of the differences

$$\exp(g_{10} + g_{01})\,h\,.\,f = f.$$

Hence the necessary and sufficient condition, that f may be a function of the differences, is the satisfaction of the linear partial differential equation

$$g_{10} + g_{01} = 0.$$

70. These operators g_{10} and g_{01} have been previously met with in the discussion of the symmetric functions connected with the fundamental identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y)\ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots,$$

but then they played a different rôle.*

* Two simple cases of this important transformation should be verified by the reader. For $n = 2$, $(\overline{11})$ is transformed into $\frac{1}{2}(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$ connected with the ternary quadric. For $n = 3$, $(\overline{21})$ is transformed into $\frac{1}{18}\Sigma(\alpha_1 - \alpha_2)\{(\alpha_1 - \alpha_3) + (\alpha_2 - \alpha_3)\}(\beta_1 - \beta_2)$ connected with the ternary cubic.

In that case g_{10} and g_{01} were shewn to annihilate all functions in which the biparts $\overline{10}$, $\overline{01}$, respectively, were absent. Hence, expressing all such functions in terms of $a_{10}, a_{01}, \ldots a_{pq}, \ldots$ we have at once a number of symmetric functions of difference of the quantities in the identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_n x + \beta_n y)$$
$$= 1 + na_{10}x + na_{01}y + \ldots + \frac{n!}{(n-p-q)!} a_{pq} x^p y^q .$$

71. The differential equation

$$g_{10} + g_{01} = 0,$$

is, as a particular case, satisfied by the solutions of the simultaneous equations

$$g_{10} = 0, \quad g_{01} = 0.$$

In correspondence therewith we have functions composed of differences $\alpha_s - \alpha_t$, $\beta_s - \beta_t$, but not of differences $\alpha_s - \beta_s$, $\alpha_s - \beta_t$. The functions of differences $\alpha_s - \alpha_t$, $\beta_s - \beta_t$ are represented by the infinite series of monomial symmetric functions whose partitions contain neither of the biparts $\overline{10}$, $\overline{01}$.

The generating function for the number of biweight pq is

$$\frac{1}{(1-x^2)(1-xy)(1-y^2)(1-x^3)(1-x^2y)(1-xy^2)(1-y^3)\ldots}.$$

The remaining functions of differences correspond to those solutions of $g_{10} + g_{01} = 0$ which are not simultaneous solutions of $g_{10} = 0$ and $g_{01} = 0$.

Denoting by N any aggregate of biparts from which both $\overline{10}$ and $\overline{01}$ are excluded, we have the system of solutions

$$(\overline{10}\,\mathrm{N}) - (\overline{01}\,\mathrm{N})$$
$$(\overline{10}^2\,\mathrm{N}) - (\overline{10}\,\overline{01}\,\mathrm{N}) + (\overline{01}^2\,\mathrm{N}),$$
$$\cdots\cdots\cdots$$
$$(\overline{10}^{p+q}\,\mathrm{N}) - (\overline{10}^{p+q-1}\,\overline{01}\,\mathrm{N}) + \ldots + (-)^q(\overline{10}^p\,\overline{01}^q\,\mathrm{N}) + \ldots,$$

for on operating with $g_{10} + g_{01} = \mathrm{G}_{10} + \mathrm{G}_{01}$ the terms destroy each other in pairs.

Observe that these solutions are of the same weight but not of the same biweight in every term.

The number of solutions of a given weight is given by the generating function

$$\frac{x}{(1-x)(1-x^2)^3(1-x^3)^4\ldots(1-x^\mu)^{\mu+1}\ldots}.$$

Hence the *whole* number of asyzygetic functions of differences of a given weight is given by

$$\frac{1}{(1-x)(1-x^2)^3(1-x^3)^4\dots(1-x^\mu)^{\mu+1}\dots}.$$

§ 13. *Special Fundamental Identity of Finite Order.*

72. By taking the fundamental identity of infinite order syzygetic relations between monomial symmetric functions were avoided. Whenever the fundamental identity is taken of a finite order > 1 certain such relations of necessity arise.

Professor CAYLEY ('Collected Papers,' vol. 2, p. 454, and 'Phil. Trans.,' 1857) takes a fundamental identity equivalent to

$$(1+\alpha_1 x+\beta_1 y)(1+\alpha_2 x+\beta_2 y)=1+2hx+2gy^2+bx^2+2fxy+cy^2,$$

and finds identically

$$bc-f^2-bg^2-ch^2+2fgh=0,$$

the condition that the expression to the right shall break up into two linear factors.

I take as the fundamental identity

$$(1+a_1 x+\beta_1 y)(1+\alpha_2 x+\beta_2 y)=1+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2,$$

and observe that the syzygetic relation must connect monomial symmetric functions, each of which is symbolised by a partition containing more than two biparts. The symmetric functions must be of the same biweight of the form pp since the quantities α must occur symmetrically with the quantities β. Of the biweight 1, 1, there exists no partition containing more than two biparts. Of the biweight 2,2, we have the four partitions

$$(\overline{20}\,\overline{01}^2),\ (\overline{02}\,\overline{10}^2),\ (\overline{11}\,\overline{10}\,\overline{01}),\ (\overline{10}^2\,\overline{01}^2),$$

and if the corresponding symmetric functions can be linearly connected, so that no fundamental symmetric function of weight greater than 2 occurs, the linear function must vanish.

From the tables, *post* § 14, biweight 22, partition $(\overline{10}^2\,\overline{01}^2)$, we find

$$(\overline{20}\,\overline{01}^2)-(\overline{11}\,\overline{10}\,\overline{01})+(\overline{02}\,\overline{10}^2)=-4a_{20}a_{02}+a_{11}{}^2+a_{20}a_{01}{}^2+a_{02}a_{10}{}^2-a_{11}a_{10}a_{01};$$

the terms involving a_{22}, a_{21}, and a_{12} disappearing.

73. This is right, and shows (changing sign) that the well-known expression (discriminant)

$$bc - f^2 - bg^2 - ch^2 + 2fgh,$$

is equal to

$$-(\overline{20}\,\overline{01}^2) + (\overline{11}\,\overline{10}\,\overline{01}) - (\overline{02}\,\overline{10}^2),$$

a form which, for the ternary quadric, vanishes at sight.

Another form is

$$-s_{20}a_{02} + s_{11}a_{11} - s_{02}a_{20}.$$

74. The expression

$$4a_{20}a_{02} - a_{11}^{\ 2} - a_{20}a_{01}^{\ 2} - a_{02}a_{10}^{\ 2} + a_{11}a_{10}a_{01}$$

satisfies the partial differential equation which appertains to the differences of the quantities in the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2.$$

This equation is

$$2\,\partial_{a_{10}} + a_{10}\,\partial_{a_{20}} + a_{01}\,\partial_{a_{11}} + 2\,\partial_{a_{01}} + a_{10}\,\partial_{a_{11}} + a_{01}\,\partial_{a_{02}} = 0.$$

It is not, as a fact, expressible as a function of differences of α_1, β_1, α_2, β_2, because it vanishes altogether for a fundamental identity of the order 2.

In relation to a fundamental identity of order greater than 2, the expression does not satisfy the equation of differences. Although it may be regarded from the above as a vanishing function of the differences, it is convertible into a non-vanishing function by the transformation before given. The transformed expression is

$$(\overline{20}\,\overline{02}) - 2(\overline{11}^2) \quad \text{or} \quad (s_{20}s_{02} - s_{11}^{\ 2}),$$

which visibly satisfies the differential equation

$$g_{10} + g_{01} \equiv \partial_{s_{10}} + \partial_{s_{01}} = 0.$$

§ 14. *The Construction of Symmetrical Tables.*

75. From the first law of symmetry it has been established that it is possible to form two symmetrical tables in connexion with every partition of every biweight. As illustrations, I give certain of the results as far as weight 4, inclusive. We have presented for the weight 4 the biweights 40, 31, 22, 13, 04. The theory of the biweights 40 and 04 is precisely the same as that of the weight 4 in the unipartite theory.* The one is, in fact, concerned only with the single system of quantities

* *Vide* 'American Journal of Mathematics,' vol. 11, and succeeding volumes.

$\alpha_1, \alpha_2, \alpha_3, \ldots$ and the other only with the single system $\beta_1, \beta_2, \beta_3, \ldots$ We may, therefore, suppress altogether the zero elements in the biparts and then proceed to form the tables for the several partitions (unipart) of the weight which I have already set forth in the 'American Journal of Mathematics' (vol. 11).

Of the remaining biweights 31, 22, 13, it is merely necessary to calculate the two former, since the tables for the biweight 13 are obviously immediately obtainable from those of the biweight 31 by interchanging the elements of each bipart :—*e.g.*, by writing $\overline{qp}$ for $\overline{pq}$.

There are seven partitions of biweight 31, viz. :—

$$(\overline{10}^3\,\overline{01}),\ (\overline{11}\,\overline{10}^2),\ (\overline{20}\,\overline{10}\,\overline{01}),\ (\overline{20}\,\overline{11}),\ (\overline{21}\,\overline{10}),\ (\overline{30}\,\overline{01}),\ (\overline{31}),$$

and nine of the biweight 22, viz. :—

$$(\overline{10}^2\,\overline{01}^2),\ (\overline{11}\,\overline{10}\,\overline{01}),\ (\overline{11}^2),\ (\overline{02}\,\overline{10}^2),\ (\overline{20}\,\overline{02}),\ (\overline{12}\,\overline{10}),\ (\overline{22}).$$
$$(\overline{20}\,\overline{01}^2),\qquad (\overline{21}\,\overline{01}),$$

Of these the table of $(\overline{20}\,\overline{01}^2)$ gives also the table of $(\overline{02}\,\overline{10}^2)$ by transposing the elements of the biparts, and similarly the table of $(\overline{21}\,\overline{01})$ gives that of $(\overline{12}\,\overline{10})$. We have thus 28 tables; but of these, the four corresponding to the partitions $(\overline{31})$ and $(\overline{22})$ are mere identities, so that the number is reduced to 24. The earlier tables which are necessary are those of the partitions $(\overline{10}\,\overline{01})$, $(\overline{20}\,\overline{01})$, $(\overline{11}\,\overline{10})$, $(\overline{10}^2\,\overline{01})$. These are now given. Each table is read according to the lines.

Biweight 11.

Partition $(\overline{10}\,\overline{01})$.

	$(\overline{10}\,\overline{01})$	$(\overline{10})\,(\overline{01})$	
$(\overline{11})$	-1	1	± 1
$(\overline{10}\,\overline{01})$	1		

	$(\overline{11})$	$(\overline{10}\,\overline{01})$
$(\overline{10}\,\overline{01})$		1
$(\overline{10})\,(\overline{01})$	1	1

Biweight 21.

Partition $(\overline{20}\,\overline{01})$.

	$(\overline{20}\,\overline{01})$	$(\overline{20})\,(\overline{01})$	
$(\overline{21})$	-1	1	± 1
$(\overline{20}\,\overline{01})$	1		

	$(\overline{21})$	$(\overline{20}\,\overline{01})$
$(\overline{20}\,\overline{01})$		1
$(\overline{20})\,(\overline{01})$	1	1

Biweight 21.

Partition $(\overline{11}\;\overline{01})$.

	$(\overline{11}\;\overline{10})$	$(\overline{11})\,(\overline{10})$	
$(\overline{21})$	-1	1	± 1
$(\overline{11}\;\overline{10})$	1		

	$(\overline{21})$	$(\overline{11}\;\overline{10})$
$(\overline{11}\;\overline{10})$		1
$(\overline{11})\,(\overline{10})$	1	1

Biweight 21.

Partition $(\overline{10}^2\;\overline{01})$.

	a_{21}	$a_{20}a_{01}$	$a_{11}a_{10}$	$a_{10}{}^2\,a_{01}$	
$(\overline{21})$	1	-1	-1	1	± 2
$(\overline{20}\;\overline{01})$	-1	-1	1		
$(\overline{11}\;\overline{10})$	-1	1			± 1
$(\overline{10}^2\;\overline{01})$	1				

	$(\overline{21})$	$(\overline{20}\;\overline{01})$	$(\overline{11}\;\overline{10})$	$(\overline{10}^2\overline{01})$
a_{21}				1
$a_{20}a_{01}$			1	1
$a_{11}a_{10}$		1	1	2
$a_{10}{}^2\,a_{01}$	1	1	2	2

Biweight 31.

Partition $(\overline{30}\;\overline{01})$.

	$(\overline{30}\;\overline{01})$	$(\overline{30})\,(\overline{01})$	
$(\overline{31})$	-1	1	± 1
$(\overline{30}\;\overline{01})$	1		

	$(\overline{31})$	$(\overline{30}\;\overline{01})$
$(\overline{30}\;\overline{01})$		1
$(\overline{30})\,(\overline{01})$	1	1

Partition $(\overline{20}\;\overline{10})$.

	$(\overline{21}\;\overline{10})$	$(\overline{21})\,(\overline{10})$	
$(\overline{21})$	-1	1	± 1
$(\overline{21}\;\overline{10})$	1		

	$(\overline{31})$	$(\overline{21}\;\overline{10})$
$(\overline{21}\;\overline{10})$		1
$(\overline{21})\,(\overline{10})$	1	1

Partition $(\overline{20}\;\overline{11})$.

	$(\overline{20}\;\overline{11})$	$(\overline{20})\,(\overline{11})$	
$(\overline{31})$	-1	1	± 1.
$(\overline{20}\;\overline{11})$	1		

	$(\overline{31})$	$(\overline{20}\;\overline{11})$
$(\overline{20}\;\overline{11})$		1
$(\overline{20})\,(\overline{11})$	1	1

Partition $(\overline{20}\,\overline{10}\,\overline{01})$.

	$(\overline{20}\,\overline{10}\,\overline{01})$	$(\overline{20}\,\overline{10})(\overline{01})$	$(\overline{20}\,\overline{01})(\overline{10})$	$(\overline{10}\,\overline{01})(\overline{20})$	$(\overline{20})(\overline{10})(\overline{01})$	
$(\overline{31})$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$\pm 1\frac{1}{2}$
$(\overline{30}\,\overline{01})$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		± 1
$(\overline{21}\,\overline{10})$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$		± 1
$(\overline{20}\,\overline{11})$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$		± 1
$(\overline{20}\,\overline{10}\,\overline{01})$	1					

	$(\overline{31})$	$(\overline{30}\,\overline{01})$	$(\overline{21}\,\overline{10})$	$(\overline{20}\,\overline{11})$	$(\overline{20}\,\overline{10}\,\overline{01})$
$(\overline{20}\,\overline{10}\,\overline{01})$					1
$(\overline{20}\,\overline{10})(\overline{01})$			1	1	1
$(\overline{20}\,\overline{01})(\overline{10})$		1		1	1
$(\overline{10}\,\overline{01})(\overline{20})$		1	1		1
$(\overline{20})(\overline{10})(\overline{01})$	1	1	1	1	1

Partition $(\overline{11}\,\overline{10}^2)$.

	$(\overline{11}\,\overline{10}^2)$	$(\overline{10}^2)(\overline{11})$	$(\overline{11}\,\overline{10})(\overline{10})$	$(\overline{11})(\overline{10})^2$	
$(\overline{31})$	1	-1	-1	1	± 2
$(\overline{20}\,\overline{11})$	-1	-1	1		
$(\overline{21}\,\overline{10})$	-1	1			± 1
$(\overline{11}\,\overline{10}^2)$	1				

	$(\overline{31})$	$(\overline{20}\,\overline{11})$	$(\overline{21}\,\overline{10})$	$(\overline{11}\,\overline{10}^2)$
$(\overline{11}\,\overline{10}^2)$				1
$(\overline{10}^2)(\overline{11})$			1	1
$(\overline{11}\,\overline{10})(\overline{10})$		1	1	2
$(\overline{11})(\overline{10})^2$	1	1	2	2

Partition $(\overline{10}^3\ \overline{01})$.

	a_{31}	$a_{30}a_{01}$	$a_{21}a_{10}$	$a_{20}a_{11}$	$a_{20}a_{10}a_{01}$	$a_{11}a_{10}^2$	$a_{10}^3a_{01}$	
$(\overline{31})$	− 1	1	1	1	− 2	− 1	1	± 4
$(\overline{30}\ \overline{01})$	1	2	− 1	− 1	− 1	1		
$(\overline{21}\ \overline{10})$	1	− 1	0	− 1	1			± 2
$(\overline{20}\ \overline{11})$	1	− 1	− 1	1				± 2
$(\overline{20}\ \overline{10}\ \overline{01})$	− 2	− 1	1					
$(\overline{11}\ \overline{10}^2)$	− 1	1						± 1
$(\overline{10}^3\ \overline{01})$	1							

	$(\overline{31})$	$(\overline{30}\ \overline{01})$	$(\overline{21}\ \overline{10})$	$(\overline{20}\ \overline{11})$	$(\overline{20}\ \overline{10}\ \overline{01})$	$(\overline{11}\ \overline{10}^2)$	$(\overline{10}^3\ \overline{01})$
a_{31}							1
$a_{30}a_{01}$						1	1
$a_{21}a_{10}$					1	1	3
$a_{20}a_{11}$				1	1	2	3
$a_{20}a_{10}a_{01}$			1	1	1	3	3
$a_{11}a_{10}^2$		1	1	2	3	4	6
$a_{10}^3a_{01}$	1	1	3	3	3	6	6

Biweight 22.

Partition $(\overline{21}\ \overline{01})$.

	$(\overline{21}\ \overline{01})$	$(\overline{21})\ (\overline{01})$	
$(\overline{22})$	− 1	1	± 1
$(\overline{21}\ \overline{01})$	1		

	$(\overline{22})$	$(\overline{21}\ \overline{01})$
$(\overline{21}\ \overline{01})$		1
$(\overline{21})\ (\overline{01})$	1	1

Partition $(\overline{20}\ \overline{02})$.

	$(\overline{20}\ \overline{02})$	$(\overline{20})\ (\overline{02})$	
$(\overline{22})$	-1	1	± 1
$(\overline{20}\ \overline{02})$	1		

	$(\overline{22})$	$(\overline{20}\ \overline{02})$
$(\overline{20}\ \overline{02})$		1
$(\overline{20})\ (\overline{02})$	1	1

Partition $(\overline{11^2})$.

	$(\overline{11^2})$	$(\overline{11})^2$
$(\overline{22})$	-2	1
$(\overline{11^2})$	1	

	$(\overline{22})$	$(\overline{11^2})$
$(\overline{11^2})$		1
$(\overline{11})^2$	1	2

Partition $(\overline{20}\ \overline{01^2})$.

	$(\overline{20}\ \overline{01^2})$	$(\overline{20})\ (\overline{01^2})$	$(\overline{20}\ \overline{01})\ (\overline{01})$	$(\overline{20})\ (\overline{01})^2$	
$(\overline{22})$	1	-1	-1	1	± 2
$(\overline{20}\ \overline{02})$	-1	-1	1		
$(\overline{21}\ \overline{01})$	-1	1			± 1
$(\overline{20}\ \overline{01^2})$	1				

	$(\overline{22})$	$(\overline{20}\ \overline{02})$	$(\overline{21}\ \overline{01})$	$(\overline{20}\ \overline{01^2})$
$(\overline{20}\ \overline{01^2})$				1
$(\overline{20})\ (\overline{01^2})$			1	1
$(\overline{20}\ \overline{01})\ (\overline{01})$		1	1	2
$(\overline{20})\ (\overline{01})^2$	1	1	2	2

Partition $(\overline{11}\,\overline{10}\,\overline{01})$.

	$(\overline{11}\,\overline{10}\,\overline{01})$	$(\overline{11}\,\overline{10})(\overline{01})$	$(\overline{11}\,\overline{01})(\overline{10})$	$2(\overline{10}\,\overline{01})(\overline{11})$	$(\overline{11})(\overline{10})(\overline{01})$	
$(\overline{22})$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	1	$\pm\frac{3}{2}$
$(\overline{21}\,\overline{01})$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$		± 1
$(\overline{12}\,\overline{10})$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$		± 1
$(\overline{11^2})$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$		$\pm\frac{1}{2}$
$(\overline{11}\,\overline{10}\,\overline{01})$	1					

	$(\overline{22})$	$(\overline{21}\,\overline{01})$	$(\overline{12}\,\overline{10})$	$(\overline{11^2})$	$(\overline{11}\,\overline{10}\,\overline{01})$
$(\overline{11}\,\overline{10}\,\overline{01})$					1
$(\overline{11}\,\overline{10})(\overline{01})$			1	2	1
$(\overline{11}\,\overline{01})(\overline{10})$		1	0	2	1
$2(\overline{10}\,\overline{01})(\overline{11})$		2	2	0	2
$(\overline{11})(\overline{10})(\overline{01})$	1	1	1	2	1

Partition $(\overline{10}^2\,\overline{01}^2)$.

	a_{22}	$a_{21}a_{01}$	$a_{12}a_{10}$	$a_{20}a_{02}$	a_{11}^2	$a_{20}a_{01}^2$	$a_{02}a_{10}^2$	$a_{11}a_{10}a_{01}$	$a_{10}^2a_{01}^2$	
$(\overline{22})$	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	1	$\pm\frac{10}{3}$
$(\overline{21}\,\overline{01})$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$		± 2
$(\overline{12}\,\overline{01})$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$		± 2
$(\overline{20}\,\overline{02})$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$		
$(\overline{11}^2)$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		$\pm\frac{4}{3}$
$(\overline{20}\,\overline{01}^2)$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		
$(\overline{02}\,\overline{10}^2)$	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		
$(\overline{11}\,\overline{10}\,\overline{01})$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$		$\pm\frac{7}{3}$
$(\overline{10}^2\,\overline{01}^2)$	1									

	$(\overline{22})$	$(\overline{21}\,\overline{01})$	$(\overline{12}\,\overline{10})$	$(\overline{20}\,\overline{02})$	$(\overline{11}^2)$	$(\overline{20}\,\overline{01}^2)$	$(\overline{02}\,\overline{10}^2)$	$(\overline{11}\,\overline{10}\,\overline{01})$	$(\overline{10}^2\,\overline{01}^2)$
a_{22}									1
$a_{21}a_{01}$							1	1	2
$a_{12}a_{10}$						1	0	1	2
$a_{20}a_{02}$					1	0	0	1	1
a_{11}^2				1	2	2	2	2	4
$a_{20}a_{01}^2$			1	0	2	0	1	2	2
$a_{02}a_{10}^2$		1	0	0	2	1	0	2	2
$a_{11}a_{10}a_{01}$		1	1	1	2	2	2	3	4
$a_{10}^2a_{01}^2$	1	2	2	1	4	2	2	4	4

§ 15. *Property of the Coefficients in the Tables.*

76. There is in regard to the coefficients a very simple and important property which does not come into view with the unipartite theory so long as the tables are restricted to the particular cases in which the separable partitions are composed entirely of units. The property appears the instant we consider a separable partition composed of parts which are *not* all similar. The law is the same whether the symmetric functions are unipartite, bipartite, or in general m-partite. It depends upon the possibility of grouping the various separations in a particular manner. To make this clear suppose we are presented with a separable partition $(\overline{10}^2\ \overline{01}^2)$. The nine separations may be written down in four groups, as follows :—

Group 1.	Group 2.	Group 3.	Group 4.
$(\overline{10})^2\ (\overline{01})^2$	$(\overline{10}\ \overline{01}^2)\ (\overline{10})$	$(\overline{10}^2\ \overline{01})\ (\overline{01})$	$(\overline{10}^2\ \overline{01}^2)$
$(\overline{11})\ (\overline{10})\ (\overline{01})$	$(\overline{01}^2)\ (\overline{10})^2$	$(\overline{10}^2)\ (\overline{01})^2$	$(\overline{10}^2)\ (\overline{01}^2)$
$(\overline{11})^2$			

In Group 2 it will be seen that the parts $(\overline{10}^2)$ of the separable partition occur in the separation $(\overline{10})^2$, while the parts $(\overline{01}^2)$ occur in the separation $(\overline{01}^2)$, so that the expression $\{(\overline{10})^2, (\overline{01}^2)\}$ may be taken as defining a certain separation property of the separations of the group. The group in question may be denoted by $\mathrm{G}r\ \{(\overline{10})^2, (\overline{01}^2)\}$ and on the same principle the Groups 1, 3, and 4 may be denoted by

$$\mathrm{G}r\ \{(\overline{10})^2, (\overline{01})^2\}, \qquad \mathrm{G}r\ \{(\overline{10}^2), (\overline{01})^2\}, \qquad \mathrm{G}r\ \{(\overline{10}^2), (\overline{01}^2)\}$$

respectively. In the separable partition $(\overline{10}^2\ \overline{01}^2)$ the parts $(\overline{10})$ and $(\overline{01})$ occur each twice, and a group results from every combination of a partition of 2 with a partition of 2. If P_2 denote the number of partitions of 2, the number of groups will be $\mathrm{P}_2{}^2 = 4$. In general if the different parts of the separable partition occur a, b c, ... times the number of groups of separations is $\mathrm{P}_a \mathrm{P}_b \mathrm{P}_c \ldots$.

77. The leading property that has been adverted to is that in the expression of a single-bipart function by means of separations of a partition composed of dissimilar parts, the algebraic sum of the coefficients in each group of separations is zero. A corollary at once follows which will be given in its proper place.

78. From the identity of Art. 24, viz. :—

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots = \Pi_s\,(1 + \alpha_s b_{10}\xi + \ldots + \alpha_s{}^p\beta_s{}^q b_{pq}\xi^p\eta^q + \ldots)$$

is derived a series of relations which express the quantities c in terms of the quantities b and symmetric functions of the quantities α, β. These are given in Art. 24.

To put the *group* in evidence it is necessary to modify these relations by writing $b_{\overline{r_1 s_1}^{\rho_1}}$ for $\overset{\rho_1}{b}_{r_1 s_1}$, so that for examples the expressions for c_{20} and c_{11} become

$$(\overline{20})\, b_{20} + (\overline{10}^2)\, b_{\overline{10}^2}$$
$$(\overline{11})\, b_{11} + (\overline{10}\,\overline{01})\, b_{10} b_{01}$$

respectively. In any product $c_{p_1 q_1}{}^{\pi_1} c_{p_2 q_2}{}^{\pi_2} \ldots$ the cofactor of the product

$$\overset{\sigma_1}{b}_{\overline{r_1 s_1}^{\rho_1}} \overset{\sigma_2}{b}_{\overline{r_1 s_1}^{\rho_2}} \ldots \overset{v_1}{b}_{\overline{t_1 u_1}^{\tau_1}} \overset{v_2}{b}_{\overline{t_1 u_1}^{\tau_2}},$$

is composed of symmetric function products each of which appertains to the group

$$Gr\ \{(\overline{r_1 s_1}^{\rho_1})^{\sigma_1} (\overline{r_1 s_1}^{\rho_2})^{\sigma_2} \ldots, (\overline{t_1 u_1}^{\tau_1})^{v_1} (\overline{t_1 u_1}^{\tau_2})^{v_2} \ldots\}.$$

The sum of the coefficients attached to the members of this group is obtained by putting each monomial symmetric function equal to unity. The sum in question then appears as the numerical coefficient of the b product above written.

Write then

$$c_{10}{}^1 = b_{10}$$
$$c_{01}{}^1 = b_{01}$$
$$c_{20}{}^1 = b_{20} + b_{\overline{10}^2}$$
$$c_{11}{}^1 = b_{11} + b_{10} b_{01}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

so that, ξ and η being arbitrary,

$$1 + c_{10}{}^1 \xi + c_{01}{}^1 \eta + \ldots + c_{pq}{}^1 \xi^p \eta^q + \ldots$$
$$= (1 + b_{10}\xi + b_{\overline{10}^2}\xi^2 + b_{\overline{10}^3}\xi^3 + \ldots) \ldots (1 + b_{pq}\xi^p\eta^q + b_{\overline{pq}^2}\xi^{2p}\eta^{2q} + \ldots) \ldots$$

a factor appearing on the right for each biweight.

To find the sum of the coefficients in each group in the case of the expression of the single-bipart functions we have now merely to take logarithms when (Art. 26) the functions being s_{pq} or $(\overline{pq})$, the sum in each case presents itself multiplied by $(-)^{p+q-1} (1/p!\,q!)\,(p+q-1)!$. Expanding the right hand side after taking logarithms we see that only terms of the form

$$\overset{\sigma_1}{b}_{\overline{rs}^{\rho_1}} \overset{\sigma_2}{b}_{\overline{rs}^{\rho_2}} \ldots$$

can appear. Hence the theorem :—

"In the expression of symmetric function $(\overline{pq})$ by means of separations of any partition of the same biweight, the partition consisting of dissimilar parts, the algebraic sum of the coefficients in each group of separations is zero."

As regards the remaining cases where the separable partition does not contain dissimilar parts, the *group* obviously contains but a single separation and *quâ group* has no existence.

We have in fact the expression of $(\overline{pq})$ by means of separations of $(\overline{rs}^k)$ where $kr = p$, $ks = q$.

The result is clearly

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}(\overline{pq}) = \Sigma\,(-)^{\Sigma\sigma-1}\frac{(\Sigma\sigma-1)!}{\sigma_1!\,\sigma_2!\ldots}(\overline{rs}^{\rho_1})^{\sigma_1}(\overline{rs}^{\rho_2})^{\sigma_2}\ldots$$

79. The law of group of separations may be verified from the tables. It is a very satisfactory aid to calculation, particularly in the detection of missing separations.

Moreover the law embraces symmetric functions other than those symbolised by a single bipart. Suppose the function expressed in terms of single-bipart functions. The latter may be separately expressed in terms of separations of partitions in such wise that the function in question will be represented by means of separations of any given partition of its biweight. The law of the group will hold for the single-bipart functions whenever the separable partition contains dissimilar parts, and moreover, in a product of single-bipart functions the law will hold if one or more of the factors is expressed in terms of separations of a partition containing dissimilar parts. Hence the only exception occurs when we find presented a product of the form

$$s_{(\overline{r_1 s_1}^{\rho_1})}\, s_{(\overline{r_2 s_2}^{\rho_2})}\, s_{(\overline{r_3 s_3}^{\rho_3})}\ldots;$$

now if the symmetric function, say $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$, whose expression we are considering in connection with a given separable partition, say $(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots)$, itself possesses a separation of specification

$$(\overline{\alpha_1 a_1,\,\alpha_1 b_1}\;\overline{\alpha_2 a_2,\,\alpha_2 b_2},\ldots)$$

a product of this form will certainly occur, but not otherwise.

Hence the theorem :—

80. "In the expression of symmetric function

$$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots),$$

by means of separations of

$$(\overline{a_1b_1}^{\alpha_1}\,\overline{a_2b_2}^{\alpha_2}\ldots),$$

the algebraic sum of the coefficients in each group of separations is zero if the partition

$(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$ possesses no separations of specification $(\overline{\alpha_1a_1, \alpha_1b_1}, \overline{\alpha_2a_2, \alpha_2b_2}\ldots)$ but not otherwise."

The law may be verified in the case of the table of separations of $(\overline{10}^2\,\overline{01}^2)$, for the symmetric functions $(\overline{22})$, $(\overline{21}\,\overline{01})$, $(\overline{12}\,\overline{10})$, $(\overline{11}^2)$, $(\overline{11}\,\overline{10}\,\overline{01})$ for none of these five functions can be separated so that the specification is $(\overline{20}\,\overline{02})$. On the other hand the group law does not hold for $(\overline{20}\,\overline{01}^2)$ because the separation $(20)\,(\overline{01}^2)$ has a specification $(\overline{20}\,\overline{02})$.

§ 16. *Conclusion.*

81. All the preceding results can be easily extended to the m-partite theory connected with m systems. The weights are m-partite as also the parts of the partitions. As a general rule m suffices appear in the symbols. The laws of symmetry and their consequences, the symmetrical tables, the correspondences between the algebras of quantity and differential operation, the partition linear and obliterating operators, the law of groups of coefficients (and in fact the whole investigation here presented) proceed *pari-passu* with the bipartite theory above set forth. The unipartite or ordinary theory of the single system is also absolutely included in every respect.

In its applications, the results will be chiefly of use in the theory of elimination in the most general case. In this regard SCHLÄFLI's memoir (*loc. cit.*) may be consulted.